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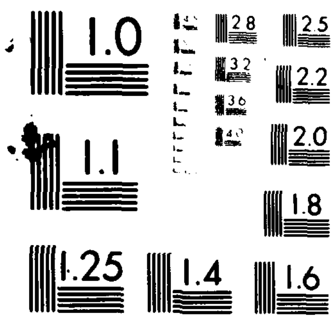

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MRC Technical Summary Report #2275

ON A NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION IN A FADING MEMORY SPACE

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September 1981

(Received June 2, 1981)

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Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

We study the linear autonomous, neutral system of functional differential equations are studied

$$(*)\qquad \begin{aligned} &\frac{d}{dt}(\mu * x(t) + f(t)) = \nu * x(t) + g(t) \quad (t > 0)\ , \\ &x(t) = \phi(t) \quad (t \le 0) \end{aligned}$$

in a fading memory space. Here μ and ν are matrix-valued measures supported on $[0,\infty)$, finite with respect to a weight function, and f, g and ϕ are C^n-valued, continuous or locally integrable functions, bounded with respect to a fading memory norm. We give conditions are given which imply that the functional differential equation solutions of ~~(*)~~ can be decomposed into a stable part and an unstable part. These conditions are of frequency domain type. ~~We~~ do not need the usual assumption that the singular part of μ vanishes. The ~~Our~~ results can be used to decompose the semigroup generated by (the functional differential equation) ~~(*)~~ into a stable part and an unstable part.

AMS (MOS) Subject Classifications: 34K20, 45F05, 45M10

Key Words: Neutral functional differential equation, fading memory space, history space, phase space, exponential dichotomy, hyperbolicity.

Work Unit Number 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

An ordinary time independent differential equation can be written as an equation where the derivative $x'(t)$ of an R^n-valued function x at time t is a given function of $x(t)$. If one allows $x'(t)$ to depend also on earlier values $x(s)$, $-\infty < s < t$, of x, then one gets a functional differential equation of retarded type. An even more general equation is obtained if one lets $x'(t)$ furthermore depend on earlier values $x'(s)$, $-\infty < s < t$ of the derivative x'. Such an equation is called a functional differential equation of neutral type.

A frequently used technique in the analysis of nonlinear differential and functional differential equations is the principle of linearization. This means that one replaces the original equation by a linear equation; one then obtains qualitative information about solutions of this linear equation, and finally one tries to show that the nonlinear equation behaves approximately in the same way as the linear equation. Typical results obtained in this way are stability theorems, bifurcation theorems, and theorems on stable and unstable manifolds. For this approach to succeed it is of crucial importance that one should obtain a fairly detailed knowledge about solutions of linear equation. Such problems arise in a variety of applications.

In this paper we study a linear system of functional differential equations of neutral type and with infinite delay. This equation can be written in the form

$$(*) \qquad \frac{d}{dt}\,(x(t) + D(x_t) + f(t)) = E(x_t) + g(t) \quad (t \geq 0) ,$$

subject to the "initial" condition $x(t) = \phi(t)$ $(t \leq 0)$. Here f and g are given vector valued input functions, $D(x_t)$ is a continuous linear functional of $x(s)$, $-\infty < s < t$ and $E(x_t)$ is a continuous linear functional of $x(s)$, $-\infty < s \leq t$. We show that under suitable assumptions motivated by applications, the fundamental solution of (*) can be split into three components: one which is exponentially stable, one which is either identically zero or exponentially unstable, and one which is either zero or gives rise to autonomous oscillations. This is exactly the type of information that one needs for attacking the associated nonlinear problems. Some earlier results of the same nature exist, but they apply only to a retarded equation (which one gets by taking $D(x_t) = 0$ in (*)), or to neutral equations with finite rather than infinite delay.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON A NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION
IN A FADING MEMORY SPACE

Olof Staffans

1. Introduction

A linear, autonomous, neutral system of functional differential equations can be written in different ways, depending on which type of solutions one is looking for. In the classical case the delay is finite, one wants the solutions to be continuous, and therefore one chooses the initial functions and the perturbation terms to be continuous. In this case one can write the equation in the form

$$\frac{d}{dt}(\mu*x(t) + f(t)) = \nu*x(t) + g(t) \quad (t \geq 0) \quad , \tag{1.1}$$

with initial condition

$$x(t) = \phi(t) \quad (t \leq 0) \quad . \tag{1.2}$$

Here f, g and ϕ are continuous, C^n-valued functions, and μ and ν are matrix-valued measures supported on a finite interval $[0,r]$. It turns out that the same formulation can be used in the case of an infinite delay, provided the measures μ and ν, this time supported on $[0,\infty)$, belong to a suitable weighted measure space, and ϕ, f and g are continuous and belong to an appropriate fading memory space.

In a recent paper [3] John A. Burns, Terry L. Herdman and Harlan W. Stech have studied the same equation in the case when the solutions belong locally to L^p instead of being continuous. In this case (1.1), (1.2) is not a well posed problem. One needs one extra piece of information to make the solution unique, i.e. one replaces (1.1) by a pair of equations

$$\mu*x(t) + f(t) = y(t) \quad (t \geq 0) \quad , \tag{1.3}$$

$$y'(t) = \nu*x(t) + g(t) \quad (t \geq 0) \quad , \tag{1.4}$$

and adds the extra initial condition

$$y(0) = y_0 \quad . \tag{1.5}$$

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A pair of functions x, y is called a solution of (1.2) - (1.5), if y is locally absolutely continuous on $[0,\infty)$ with $y(0) = y_0$, if y' and x belong locally to L^p, and (1.3) - (1.5) hold a.e. (see (2.11) below for a definition of the convolutions). The only difference compared to (1.1), (1.2), is that this time $\mu * x(t) + f(t)$ is well defined only almost everywhere, so we have to give the value $y(0)$ explicitly, instead of calculating it from (1.3). Again, the theory in [3] applies only to the case when the delay is finite, but as we shall see below, it can be extended to the case of an infinite delay.

Daniel Henry [22] has studied (1.1), (1.2) with finite delay and continuous solutions, and obtained exponential growth estimates for the solutions. Let $D(z)$ be the characteristic function of the kernel in (1.1), i.e.

$$D(z) = z\hat{\mu}(\tau) - \hat{\nu}(\tau) \quad ,$$

where $\hat{\mu}(\tau)$ and $\hat{\nu}(\tau)$ are the Laplace transforms of μ and ν. If $\det D(z) \neq 0$ on a line $\mathrm{Re}\, z = \lambda$, then there is some hope of splitting the solutions of (1.1), (1.2) into two solutions, one which grows faster than $\exp(\lambda t)$, and one which grows slower than $\exp(\lambda t)$ as $t \to \infty$. More specifically, one wants to get a decomposition $x = x_S + x_U$, where x_S is a "stable" solution of (1.1) (but does not necessarily satisfy the initial condition (1.2)), and x_U is an "unstable" solution of the homogeneous equation

$$\text{(1.6)} \qquad \frac{d}{dt}(\mu * x(t)) = \nu * x(t) \qquad (-\infty < t < \infty) \quad .$$

Observe in particular that we want x_U to satisfy (1.6) also for negative values of t. Indeed, Henry [22] succeeds to get such a decomposition, if $\inf_{\mathrm{Re}\, z = \lambda} |\det D(z)| > 0$, μ has no singular part, and $f = g = 0$ (Henry also studies the nonhomogeneous equation, and a nonlinear equation). As far as we know, the question of what happens when μ has a nonzero singular part has been open. Similar decomposition theorems have been proved by Toshiki Naito [41], [42] for the retarded equation (which one gets by replacing $\mu * x$ by x) with infinite delay in an L^p-setting, and for even more general "phase spaces". This time the "stable" solution x_S, y_S satisfies (1.3), (1.4), and the "unstable" solution x_U, y_U satisfies

$$(1.7) \qquad \mu * x(t) = y(t) \qquad (-\infty < t < \infty) \quad ,$$

$$(1.8) \qquad y'(t) = \nu * x(t) \qquad (-\infty < t < \infty) \quad .$$

Observe that if (1.1), (1.3) and (1.4) are homogeneous, i.e. $f = g = 0$, then both the stable and the unstable component satisfy the original equation, except for the initial conditions.

Here we develop a decomposition theory which applies to neutral (as well as retarded) equations with (finite or) infinite delay in a continuous and a L^p setting. We do not have to assume that the singular part of μ vanishes, although the result that we get permits a stronger decomposition when it does vanish. On the other hand, our setting is less general than e.g. the abstract phase space discussed in [20].

2. On Some Fading Memory Spaces Compatible with a Weighted Measure Space

There exists an extensive theory on spaces of fading memory type; see e.g. [4] - [9], [15], [20], [27], [34], [37], [41] and [42] ([37] contains a fairly complete discussion of the theory prior to 1977). It is a common feature in fading memory spaces that the norm of the translation operator τ_h, defined by

$$(2.1) \qquad \tau_h \phi(t) = \phi(t+h) \qquad (t,h \in R) \quad ,$$

plays a crucial role when one investigates asymptotic properties. The norm of the translation operator is a submultiplicative function, and we get a connection to the theory of weighted measures in e.g. [13], [30] by choosing the weight of [13], [30] to be essentially the norm of the translation operator. Actually, below we use this idea, but formally we proceed in a slightly different way. We first define the concept of a "dominating function" $\rho(h)$, and then introduce some fading memory spaces, in which the norm of the translation operator is dominated by $\rho(-h)$.

We call ρ a <u>dominating function on</u> R, if ρ is strictly positive, continuous, and satisfies

$$(2.2) \qquad \rho(s+t) \le \rho(s)\rho(t) \qquad (s,t \in R); \ \rho(0) = 1 \quad .$$

The condition (2.2) is the same as the condition (2.1) in [30], plus the additional $\rho(0) = 1$. We call η an <u>influence function dominated by</u> ρ, if η is continuous, strictly positive, and satisfies

$$(2.3) \qquad \eta(s+t) \leq \rho(s)\eta(t) \quad (s,t \in R); \ \eta(0) = 1 \ .$$

Observe that (2.3) implies (replace t by $s+t$ and s by $-s$)

$$(2.4) \qquad \tilde{\rho}(s)\eta(t) \leq \eta(s+t) \quad (s,t \in R) \ ,$$

where $\tilde{\rho}$ is the function

$$(2.5) \qquad \tilde{\rho}(t) = (\rho(-t))^{-1} \quad (t \in R) \ .$$

Moreover, it is easy to see that both ρ and $\tilde{\rho}$ are influence functions dominated by ρ, and that every influence function η dominated by ρ satisfies

$$(2.6) \qquad \tilde{\rho}(t) \leq \eta(t) \leq \rho(t) \quad (t \in R) \ .$$

We warn the reader that our definition of an influence function is not quite the standard one. In general one defines the functions η and ρ only on $R^- = (-\infty,0]$ or on $R^+ = [0,\infty)$ instead of R. We could do so also here, i.e., follow [4] and work with one dominating function on R^+ and another on R^-, but the present approach simplifies the basic theory considerably.

We let $M(\rho;C)$ be the set of all complex, locally finite measures on R such that

$$(2.7) \qquad \|\mu\| = \int_R \rho(t)d|\mu|(t) < \infty \ .$$

Here $|\mu|$ is the total variation (measure) of μ. Let $C^{n\times n}$ be the set of $n\times n$-dimensional complex matrices, and let $M(\rho;C^{n\times n})$ be the matrix-valued analogue of $M(\rho;C)$ (this time one uses a matrix norm when one computes the total variation $|\mu|$ of μ).

In the sequel we shall define function spaces with values in either C, C^n or $C^{n\times n}$. Most of the time it is irrelevant in which space the values of the functions lie, and therefore we do not specify this space explicitly. In the same way, it is often irrelevant in which space the values of our measures lie, and we abbreviate both $M(\rho;C)$ and $M(\rho;C^{n\times n})$ by $M(\rho)$.

For every influence function η dominated by ρ, we let $BC_0(\eta)$ be the set of continuous functions ϕ on R satisfying

$$\lim_{t\to\pm\infty} \eta(t)\phi(t) = 0 \quad ,$$

with norm

$$\|\phi\| = \sup_{t\in R} \eta(t)|\phi(t)| \quad . \tag{2.8}$$

We show below that these spaces are contained in the space $BUC(\eta)$ or "uniformly continuous" functions, defined as follows. A continuous function ϕ on R belongs to $BUC(\eta)$ if the norm in (2.8) is finite, and if it is uniformly continuous in the sense that

$$\lim_{h\to 0} \|\tau_h\phi - \phi\| = 0 \quad ,$$

where τ_h is the translation operator defined in (2.1). By (2.6), $BC_0(\rho) \subset BC_0(\eta) \subset BC_0(\tilde{\rho})$, and $BUC(\rho) \subset BUC(\eta) \subset BUC(\tilde{\rho})$.

We let $L^p(\eta)$, $1 \le p < \infty$, be the set of measurable functions on R satisfying $\|\phi\|_p < \infty$, where

$$\|\phi\|_p = (\int_R (\eta(t)|\phi(t)|)^p dt)^{1/p} \quad . \tag{2.9}$$

The space $L^\infty(\eta)$ is defined analogously, with

$$\|\phi\|_\infty = \operatorname{ess\,sup}_{t\in R} \eta(t)|\phi(t)| \quad . \tag{2.10}$$

Observe that $BUC(\eta) \subset L^\infty(\eta)$, and that for $\phi \in BUC(\eta)$, one may write $\|\phi\| = \|\phi\|_\infty$ (cf. (2.8)). Moreover $L^p(\rho) \subset L^p(\eta) \subset L^p(\tilde{\rho})$, $1 \le p \le \infty$.

One can define, at least formally, the convolution $\mu*\phi$ of a measure $\mu \in M(\rho)$ and a function $\phi \in L^p(\eta)$, $1 \le p \le \infty$, by

$$\mu*\phi(t) = \int_R d\mu(s)\phi(t-s) \quad . \tag{2.11}$$

If $\mu = \{\mu_{ij}\}$ is matrix-valued, and $\phi = \{\phi_j\}$ is vector-valued, then $\mu*\phi$ in (2.11) should be interpreted as the vector-valued function whose j^{th} component is $\sum_k \mu_{jk}*\phi_k$, with analogous definitions when μ is complex-valued and ϕ is vector-valued, etc.

<u>Lemma 2.1</u>. <u>For every</u> $\mu \in M(\rho)$, <u>the convolution operator</u> $\phi \to \mu*\phi$ <u>maps</u> $L^p(\eta)$, $1 \le p \le \infty$, $BUC(\eta)$ <u>and</u> $BC_0(\eta)$ <u>into themselves, and</u>

$$\|\mu*\phi\|_p \le \|\mu\| \, \|\phi\|_p \quad . \tag{2.12}$$

Actually, the proof given below shows that $\mu*$ also maps the subspaces $\{\phi \in BUC(\eta) \mid \eta(t)\phi(t) \to 0 \quad (t \to \infty)\}$ and $\{\phi \in BUC(\eta) \mid \eta(t)\phi(t) \to 0 \quad (t \to -\infty)\}$ of $BUC(\eta)$ into themselves.

<u>Proof</u>. The function $s \mapsto \phi(t-s)$ is measurable with respect to μ for almost all $t \in R$ [24, Theorem 13.9 and Lemma 20.7], so the convolution in (2.11) is well defined, provided we can show that

$$|\mu| * |\phi|(t) = \int_R |\phi(t-s)|d|\mu|(s)$$

is finite a.e. However, by (2.3), with t replaced by $t - s$, and (2.11) ,

$$\eta(t)|\mu*\phi|(t) \le \eta(t)|\mu|*|\phi|(t) \le (\rho|\mu|) * (\eta|\phi|)(t) \quad , \tag{2.13}$$

so (2.12) follows from e.g. [24, Theorem 20.12], and (2.11) converges almost everywhere. If $\phi \in L^\infty(\eta)$ is continuous, then (2.11) converges everywhere, and we may replace the norm in (2.12) by the norm defined in (2.8).

If $\phi \in BUC(\eta)$, then $\|\tau_h(\mu*\phi) - \mu*\phi\| = \|\mu*(\tau_h\phi - \phi)\| \le \|\mu\| \, \|\tau_h\phi - \phi\| \to 0 \ (h \to 0)$, so $\mu*\phi \in BUC(\eta)$.

In Lemma 2.2 below we prove that $BC_0(\eta) \subset BUC(\eta)$. This, combined with the previous paragraph, shows that $\mu*\phi$ is continuous whenever $\phi \in BC_0(\eta)$. Let $\phi \in BUC(\eta)$ satisfy $\eta(t)\phi(t) \to 0 \quad (t \to \infty)$. Then by (2.11) and (2.13),

$$\eta(t)|\mu*\phi(t)| \le \Big(\int_{(-\infty,t/2]} + \int_{(t/2,\infty)}\Big)\eta(t-s)|\phi(t-s)|\rho(s)d|\mu|(s)$$

$$\le \int_{(-\infty,t/2]} \rho(s)d|\mu|(s) \cdot \sup_{s \ge t/2} \eta(s)|\phi(s)|$$

$$+ \int_{(t/2,\infty)} \rho(s)d|\mu|(s) \cdot \sup_{s < t/2} \eta(s)|\phi(s)| \to 0 \quad (t \to \infty) \quad .$$

The same computation shows that $\eta(t)\mu*\phi(t) \to 0 \quad (t \to -\infty)$ whenever $\eta(t)\phi(t) \to 0 \quad (t \to -\infty)$, so $\mu*\phi \in BC_0(\eta)$ whenever $\phi \in BC_0(\eta)$.

□

<u>Lemma 2.2</u>. <u>The translation operator</u> τ_h <u>is bounded in</u> $L^p(\eta)$, $1 \le p < \infty$, <u>with</u>

$$\|\tau_h\phi\|_p \le \rho(-h)\|\phi\|_p \quad . \tag{2.14}$$

<u>It is strongly continuous in</u> $BC_0(\eta)$, <u>in</u> $BUC(\eta)$ <u>and in</u> $L^p(\eta)$ <u>for</u> $1 \le p < \infty$. <u>In particular</u>, $BC_0(\eta) \subset BUC(\eta)$.

Lemma 2.2 is essentially contained in [4, Remark 3.2].

Proof. That the translation operator is strongly continuous in $BUC(\eta)$ is built into the definition of $BUC(\eta)$. That (2.14) holds follows from (2.12) and the fact that $\tau_h\phi = \delta_{-h}*\phi$ where δ_{-h} is the unit point mass at $-h$. If ϕ is continuous and has compact support, then $\tau_h\phi \to \phi$ uniformly, and in $L^p(\eta)$ for $1 \leq p \leq \infty$. The set of continuous functions with compact support is dense in $BC_0(\eta)$ and in $L^p(\eta)$ for $1 \leq p < \infty$, and this together with the fact that the translation operator is bounded implies the strong continuity in $BC_0(\eta)$ and $L^p(\eta)$, $1 \leq p < \infty$.

□

Lemma 2.3. Let $a \in L^p(\rho)$, and $\phi \in L^q(\eta)$, where $1 \leq p \leq \infty$, and $1/p + 1/q = 1$. Then $a*\phi \in BUC(\eta)$, and

$$\|a*\phi\|_\infty \leq \|a\|_p\|\phi\|_q \quad . \tag{2.15}$$

If $1 < p < \infty$, then $a*\phi \in BC_0(\eta)$.

Proof. That (2.15) holds and that $\eta(t)a*\phi(t) \to 0$ $(t \to \pm\infty)$ when $1 < p < \infty$ follows from (2.13) and Hölder's inequality (see e.g. [24, p. 295]). The uniform continuity is due to the fact that translation is continuous in $L^p(\rho)$ for $1 \leq p < \infty$, and in $L^q(\eta)$ for $1 \leq q < \infty$.

□

We say that an influence function η has the relaxation property, if

$$\eta(t) = o(\rho(t)) \quad (t \to -\infty) \quad . \tag{2.16}$$

Obviously, the function ρ itself never has the relaxation property, and by (2.6), a necessary and sufficient condition for the existence of an influence function dominated by ρ with the relaxation property is that $\check{\rho}$ has the relaxation property. Again, we warn the reader that our definition of the relaxation property is related to the relaxation property in e.g. [4], but not identical to it. They become approximately identical if $\rho(t) \equiv 1$ for $t \leq 0$ (cf. Lemmas 2.4 and 2.6 below).

Lemma 2.4. Let η be an influence function with the relaxation property, and let $\phi \in BC_0(\eta)$, $\psi \in L^p(\eta)$, $1 \leq p < \infty$. Then $\|\tau_t\phi\| = o(\rho(-t))$ and $\|\tau_t\psi\|_p = o(\rho(-t))$ as $t \to \infty$.

Proof. We only prove the statement concerning ϕ, the proof for ψ being completely analogous.

Fix $\varepsilon > 0$. Choose T so large that

$$\sup_{|t|>T} \eta(t)|\phi(t)| < \varepsilon \quad . \tag{2.17}$$

Define

$$\beta = \sup_{-T<t<T} \rho(t)|\phi(t)| \quad , \tag{2.18}$$

and choose S so large that

$$\tilde{\rho}(t)\eta(-t) < \varepsilon/\beta \quad (t > S) \quad . \tag{2.19}$$

Use (2.3), (2.4) and (2.17) - (2.19) to get for $t > S$,

$$\tilde{\rho}(t)\|\tau_t\phi\| = \tilde{\rho}(t) \sup_{s\in R} \eta(s)|\phi(s+t)|$$

$$\leq \tilde{\rho}(t) \sup_{|s+t|>T} \eta(s)|\phi(s+t)| + \tilde{\rho}(t) \sup_{|s+t|<T} \eta(s)|\phi(s+t)|$$

$$\leq \sup_{|s+t|>T} \eta(s+t)|\phi(s+t)| + \tilde{\rho}(t)\eta(-t) \sup_{|s+t|<T} \rho(s+t)|\phi(s+t)|$$

$$\leq \varepsilon + \tilde{\rho}(t)\eta(-t)\beta < 2\varepsilon \quad .$$

This shows that $\tilde{\rho}(t)\|\tau_t\phi\| \to 0 \quad (t \to \infty)$.

□

The spaces $BC_0(\eta)$ and $L^p(\eta)$ are defined in such a way that $\phi \in BC_0(\eta)$ iff $\eta\phi \in BC_0$, where BC_0 is the set of continuous functions vanishing at $\pm\infty$, and $\phi \in L^p(\eta)$ iff $\eta\phi \in L^p$, where L^p is the standard, non-weighted L^p-space. A similar result is true for $BUC(\eta)$.

Lemma 2.5. $\phi \in BUC(\eta)$ iff $\eta\phi \in BUC$, where BUC is the set of bounded, uniformly continuous functions on R.

Proof. Clearly $\|\phi\| < \infty$ iff $\eta\phi$ is bounded, so it suffices to show that the two concepts of uniform continuity agree. We have to show that

$$\lim_{h\to 0} \sup_{t\in R} \eta(t)|\phi(t+h) - \phi(t)| = 0$$

iff

$$\lim_{h\to 0} \sup_{t\in R} |\eta(t+h)\phi(t+h) - \eta(t)\phi(t)| = 0 \quad .$$

But

$$\eta(t+h)\phi(t+h) - \eta(t)\phi(t) - \eta(t)(\phi(t+h) - \phi(t))$$

$$= (\eta(t+h) - \eta(t))\phi(t+h) \quad ,$$

so this is equivalent to

$$\lim_{h\to 0} \sup_{t\in R} |\eta(t+h) - \eta(t)|\ |\phi(t+h)| = 0 \quad .$$

By (2.3), (2.4), the continuity of ρ and $\breve{\rho}$ at zero, and the fact that $\rho(0) = \breve{\rho}(0) = 1$,

$$\lim_{h\to 0} \sup_{t\in R} \frac{|\eta(t+h)-\eta(t)|}{\eta(t+h)} = 0 \quad .$$

This means that

$$\lim_{h\to 0} \sup_{t\in R} |\eta(t+h) - \eta(t)|\ |\phi(t+h)|$$

$$\leq \|\phi\| \lim_{h\to 0} \sup_{t\in R} \frac{|\eta(t+h)-\eta(t)|}{\eta(t+h)} = 0 \quad .$$

□

When we discuss (1.1), (1.2) and (1.2) - (1.5) in a semigroup setting, we shall work in one of the spaces $BUC(\eta)$, $BC_0(\eta)$ or $L^p(\eta)$, $1 \leq p < \infty$, but restrict our functions to R^-. We denote the restricted spaces $BUC(R^-;\eta)$, $BC_0(R^-;\eta)$ and $L^p(R^-;\eta)$. One gets norms for these spaces by restricting t in (2.8), (2.9) to R^-. The spaces $BUC(R^+;\eta)$, $BC_0(R^+;\eta)$ and $L^p(R^+;\eta)$ are defined analogously.

We define the combined <u>translation and restriction operator</u> Δ_t by

(2.20) $$\Delta_t\phi(s) = \phi(s+t) \quad (s \in R^-) \quad .$$

Observe that $\Delta_t = \Delta_0\tau_t$, and that Δ_t maps $BUC(\eta)$ into $BUC(R^-;\eta)$, $BC_0(\eta)$ into $BC_0(R^-;\eta)$, and $L^p(\eta)$ into $L^p(R^-;\eta)$.

When one studies a nonlinear neutral functional differential equation (e.g. with finite delay) it is often of crucial importance to know that the trajectories of bounded solutions are relatively compact. We shall not discuss the nonlinear equation in this paper, but we want to record the following compactness result for future use.

Lemma 2.6. *Let* ρ *be a dominating function satisfying* $\rho(t) \equiv 1$ $(t \in R^-)$, *and let* η *be an influence function dominated by* ρ, *with the relaxation property. Let* V *be a set of continuous functions* ϕ *on* R *satisfying* $\Delta_t \phi \in BC_0(R^-;\eta)$ *for* $t \in R^+$. *Then the set* $\{\Delta_t \phi | \phi \in V,\ t \in R^+\}$ *is relatively compact in* $BC_0(R^-;\eta)$ *iff for every* $T > 0$, *the set* V *is uniformly bounded and equicontinuous on* $(-T,\infty)$, *and*

$$\lim_{t\to-\infty} \sup_{\phi\in V} \eta(t)|\phi(t)| = 0 \quad . \tag{2.21}$$

One could prove Lemma 2.6 by essentially repeating the argument used by Yoshiyuki Hino in the proof of his corresponding compactness lemma [27, Lemma 4]. However, we prefer to give a slightly different proof.

Proof. That (2.21) is a necessary condition for relative compactness follows from the fact that the set $\{\Delta_0 \psi | \phi \in V\}$ is totally bounded, and each $\phi \in V$ satisfies $\lim_{t\to-\infty} \eta(t)\phi(t) = 0$. On each interval $[-T,0]$, η is bounded away from zero, so relative compactness in $BC_0(R^-;\eta)$ implies relative compactness in $BC[-T,0]$, the set of bounded continuous functions on $[-T,0]$, with the maximum-norm. Thus, if $\{\Delta_t \phi | \phi \in V,\ t \in R^+\}$ is relatively compact in $BC_0(R^-;\eta)$, then by the converse of Ascoli's theorem, the set $\{\phi(s+t) | \phi \in V,\ t \in R^+\}$ is uniformly bounded and equicontinuous for $-T \le s \le 0$. But this implies that V is uniformly bounded and equicontinuous on each interval $[-T,\infty]$, as claimed.

Conversely, suppose that for each $T > 0$, V restricted to $[-T,\infty]$ is uniformly bounded and equicontinuous, and that (2.21) holds. We claim that this implies

$$\lim_{s\to-\infty} \sup_{\substack{\phi\in V \\ t\in R^+}} \eta(s)|\phi(s+t)| = 0 \quad . \tag{2.22}$$

Define

$$h(t) = \sup_{\phi\in V} \eta(t)|\phi(t)| \quad . \tag{2.23}$$

Then $h(t) \to 0 \quad (t \to -\infty)$, and

(2.24) $$\sup_{t\in R} h(t) = N < \infty \quad .$$

Put

(2.25) $$g(t) = \eta(t) \sup_{t\leq s<\infty} \frac{h(s)}{\eta(s)} \quad .$$

Then $g(t) \geq h(t) \quad (t \in R)$. Observe that by (2.3) and our assumptions on ρ, $\eta(t) \leq \rho(t-s)\eta(s) \leq \eta(s)$ for $t \leq s$, so η is nondecreasing. This fact together with (2.24) and (2.25) yields $g(t) \leq N \quad (t \in R)$. We claim that $g(t) \to 0 \quad (t \to -\infty)$. Fix $\varepsilon > 0$, and choose T so small that $h(t) \leq \varepsilon \quad (t \leq T)$. Then for $t \leq T$,

$$g(t) \leq \eta(t) \sup_{T\leq s<\infty} \frac{h(s)}{\eta(s)} + \varepsilon \quad .$$

Thus, as $\eta(t) \to 0 \quad (t \to -\infty)$, we have $g(t) \leq 2\varepsilon$ for t sufficiently small, so indeed, $g(t) \to 0 \quad (t \to -\infty)$.

By (2.23), every $\phi \in V$ satisfies

$$|\phi(s)| \leq h(s)/\eta(s) \leq g(s)/\eta(s) \quad (s \in R) \quad .$$

It follows from (2.25) that g/η is nonincreasing, and so also

$$|\tau_t\phi(s)| = |\phi(s+t)| \leq g(s+t)/\eta(s+t)$$
$$\leq g(s)/\eta(s) \quad (t \in R^+,\ s \in R) \quad .$$

Thus,

$$\sup_{\substack{\phi\in V\\ t\in R^+}} \eta(s)|\phi(s+t)| \leq g(s) \to 0 \quad (s \to -\infty) \quad .$$

This proves (2.22).

As $BC_0(R^-;\eta)$ is a Banach space, it suffices to prove that $\{\Delta_t\phi | \phi \in V,\ t \in R^+\}$ is relatively sequentially compact. Take sequences $\phi_k \in V$ and $t_k \in R^+$. By Ascoli's theorem, we can find subsequences (which we again denote by ϕ_k and t_k) and a continuous function ψ such that $\tau_{t_k}\phi_k \to \psi$ uniformly on compact subsets of R. Fix $\varepsilon > 0$. Then, because of (2.22), we can find a number $T \leq 0$ such that

$$\sup_{s\leq T} \eta(s)|\tau_{t_k}\phi_k(s)| \leq \varepsilon, \quad \sup_{s\leq T} \eta(s)|\psi(s)| \leq \varepsilon \quad .$$

On $[-T,0]$ we have uniform convergence, so for k big enough

$$\sup_{T\leq s\leq 0} \eta(s)|\tau_{t_k}\phi_k(s) - \psi(s)| \leq \varepsilon \quad .$$

This shows that $\Delta_{t_k}\phi_k \to \Delta_0\psi$ in $BC_0(R^-;\eta)$, and completes the proof of Lemma 2.6.

□

3. On Laplace Transforms and Derivatives

We recall from [30] that to every dominating function one may adjoin two real numbers ρ_* and ρ^*, $-\infty < \rho_* \leq \rho^* < \infty$, as follows

$$\text{(3.1)} \qquad \begin{aligned} \rho_* &= -\inf_{t>0} \frac{\log \rho(t)}{t} = -\lim_{t\to\infty} \frac{\log \rho(t)}{t} \quad , \\ \rho^* &= -\sup_{t<0} \frac{\log \rho(t)}{t} = -\lim_{t\to-\infty} \frac{\log \rho(t)}{t} \quad . \end{aligned}$$

Moreover,

(3.2) $\qquad M(\rho) \subset M(\exp(-\lambda t))$ <u>for every</u> λ, $\rho_* \leq \lambda \leq \rho^*$.

The bilateral Laplace transform of a measure μ if defined by

(3.3) $$\hat{\mu}(z) = \int_R e^{-zt} d\mu(t) \quad (\rho_* \leq \text{Re } z \leq \rho^*) \quad .$$

Equivalently, define $e_z(t) = \exp(zt)$, observe that $e_z \in BUC(\tilde{\rho};C)$, and define

(3.4) $$\hat{\mu}(z) = \mu * e_z(0) \quad (\rho_* \leq \text{Re } z \leq \rho^*) \quad .$$

Let q, $r \in M(\rho)$. Then the mapping $\phi \to q*(r*\phi)(0)$ is continuous form $BC_0(\tilde{\rho})$ into C (or C^n or $C^{n\times n}$), and by the Riesz representation theorem, it is induced by a measure $s \in M(\rho)$, i.e. we can find a measure $s \in M(\rho)$ such that

$$s*\phi(0) = q*(r*\phi)(0) \quad .$$

We define the convolution $q*r$ of q and r to be this measure s. Then by definition

$$(q*r)*\phi(0) = q*(r*\phi)(0) \quad ,$$

and as translation commutes with convolution, we get

(3.5) $$(q*r)*\phi = q*(r*\phi) \quad .$$

Once one knows that (3.5) holds for $\phi \in BC_0(\check{\rho})$, one can show that it must also hold for $\phi \in BUC(\eta)$, and for $\phi \in L^p(\eta)$, $1 \leq p \leq \infty$ (in an almost everywhere sense), where η is an arbitrary influence function dominated by ρ. If $\phi \in L^p(\eta)$, $1 \leq p < \infty$, then we can find $\psi \in BC_0(\check{\rho}) \cap L^p(\eta)$ such that $\|\phi-\psi\|_p$ is arbitrarily small. As (3.5) holds for ψ, and the convolution operator is continuous in $L^p(\eta)$, we get (3.5) for every $\phi \in L^p(\eta)$. If $\phi \in L^\infty(\eta)$, then we can find $\check{\phi}$ such that $\check{\phi}$ is Borel measurable, that $\|\phi-\check{\phi}\| = 0$, and $\sup_{t\in R} \eta(t)\check{\phi}(t) < \infty$. If $\phi \in BUC(\eta)$ then take $\check{\phi} = \phi$. We can then find a Borel measurable sequence $\psi_n \in L^1(\eta)$ (or $\psi_n \in BC_0(\check{\rho})$ in the continuous case) such that $\eta(t)\psi_k(t)$ is uniformly bounded, and $\psi_n(t) \to \check{\phi}(t)$ pointwise. By Lebesgue's dominated convergence theorem,

$$(q * r) * \check{\phi}(t) = \lim_{n\to\infty} (q*r) * \psi_n(t)$$

$$= \lim_{n\to\infty} q * (r * \psi_n)(t) = g * (r * \check{\phi})(t)$$

(the second equality holds a.e. in the L^∞-case, and everywhere in the continuous case. Thus, if $\phi \in BUC(\eta)$, then (3.5) holds everywhere, and if $\phi \in L^\infty(\eta)$, then (3.5) holds almost everywhere.

If one applies (3.5) with $\phi(t) = \exp(zt)$ and uses (3.4), then one gets

$$(q*r)\hat{}(z) = \hat{q}(z)\hat{r}(z) \quad .$$

In particular, the notion of convolution defined here is equivalent to the notion used in [13] and [30].

The following lemma characterizes those measures in $M(\rho)$ whose (distribution) derivative also belongs to $M(\rho)$.

Lemma 3.1. Suppose that $p, q \in M(\rho)$, and that for some $\lambda \in R$, $\rho_* < \lambda < \rho^*$,

(3.6) $$z\hat{p}(z) = \hat{q}(z) \quad (\mathrm{Re}\ z = \lambda) \quad .$$

Then p is induced by a function a, i.e. $dp(s) = a(s)ds$, which is locally of bounded variation, and whose measure derivative da equals q. In particular, $da \in M(\rho)$. Conversely, suppose that $a \in L^1(\rho)$, and $da \in M(\rho)$. Then

(3.7) $$z\hat{a}(z) = (da)\hat{}(z) \quad (\rho_* < \mathrm{Re}\ z < \rho^*) \quad .$$

Proof. Define $dp_\lambda(t) = \exp(-\lambda t)dp(t)$. Then p_λ is a bounded measure, in particular, it is a tempered distribution, whose (distribution) Fourier transform $\hat{p}_\lambda(i\omega)$ equals $\hat{p}(\lambda+i\omega)$ $(\omega \in R)$. The Fourier transform of the distribution derivative p'_λ of p_λ equals $i\omega\hat{p}(\lambda+i\omega)$ $(\omega \in R)$, so the transform of $\lambda p_\lambda + p'_\lambda$ equals $(\lambda+i\omega)\hat{p}(\lambda+i\omega)$ $(\omega \in R)$. On the other hand, this is also the transform of q_λ, defined by $dq_\lambda(t) = \exp(-\lambda t)dq(t)$. Thus, $\lambda p_\lambda + p'_\lambda = q_\lambda$. In particular, $p'_\lambda = q_\lambda - \lambda p_\lambda$ is a finite measure, hence p_λ is induced by a function α_λ of bounded variation. This means that p is induced by a function a which is locally of bounded variation, and

$$\lambda \exp(-\lambda t)a(t)dt + d(\exp(-\lambda t)a(t)) = \exp(-\lambda)dq(t) \quad .$$

But $d(\exp(-\lambda t)a(t)) = -\lambda \exp(-\lambda t)a(t)dt + \exp(-\lambda t)da(t)$, so we get $da(t) = dq(t)$.

The converse statement is proved in a similar way. One gets (3.7) in the distribution sense on all lines of the form $\operatorname{Re} z = \lambda$, $\rho_* < \lambda < \rho^*$, even without the assumption $da \in M(\rho)$. As $da \subset M(\rho)$, we know that $(da)\hat{}(z)$ is a continuous function, and this makes (3.7) hold in the classical sense.

□

Lemma 3.2. *Let* $\mu \in M(\rho)$, *and let* $a \in L^1(\rho)$ *with* $da \in M(\rho)$. *Then* $\mu * a$ *is locally of bounded variation, and* $d(\mu * a) = \mu * da \in M(\rho)$.

Proof. Fix any λ, $\rho_* < \lambda < \rho^*$. By Lemma 3.1, on the line $\operatorname{Re} z = \lambda$,

$$\begin{aligned} z(\mu*a)\hat{}(z) &= z\hat{\mu}(z)\hat{a}(z) = \hat{\mu}(z)(z\hat{a}(z)) = \hat{\mu}(z)(da)\hat{}(z) \\ &= (\mu*da)\hat{}(z) \quad . \end{aligned}$$

Thus, by Lemma 3.1, $\mu * a$ is locally of bounded variation, and $d(\mu * a) = \mu * da$.

□

The following lemma plays a crucial role in our study of the differentiability properties of functions in $BUC(\eta)$ and $L^p(\eta)$, $1 \le p < \infty$.

Lemma 3.3. *Define* $\delta_h(t) = 1/h$ $(-h \le t \le 0)$ *if* $h > 0$, $\delta_h(t) = -1/h$ $(0 \le t \le -h)$ *if* $h < 0$, *and* $\delta_h(t) = 0$ *otherwise*. *If* $\phi \in L^p(\eta)$, $1 \le p < \infty$, *then* $\delta_h * \phi \to \phi$ *in* $L^p(\eta)$ *as* $h \to 0$. *The same statement is true with* $L^p(\eta)$ *replaced by* $BUC(\eta)$, *and by* $BC_0(\eta)$.

The L^p version of Lemma 3.3 is essentially contained in [37, Lemma 2.4, p. 73]. As $h \to 0$, once can regard δ_h as an approximation of the unit point mass δ at zero. The proof of Lemma 3.3 given below could easily be extended to other "approximate identities" than δ_h.

<u>Proof</u>. First consider the case $\phi \in BUC(\eta)$. We have to show that

$$\eta(t)\ \left|\frac{1}{h}\int_{-h}^{0} \phi(t-s)ds - \phi(t)\right|$$

tends to zero as $h \to 0$, uniformly in t. Write this expression as

$$\eta(t)\ \frac{1}{|h|}\ \left|\int_t^{t+h} (\phi(s) - \phi(t))ds\right|$$

$$\leq \eta(t)\ \frac{1}{|h|}\ \left|\int_t^{t+h} |\phi(s) - \phi(t)|ds\right| \quad .$$

As $\phi \in BUC(\eta)$, $\|\tau_h\phi - \phi\| \to 0$ $(h \to 0)$, which means that for every $\varepsilon > 0$, we can find $\gamma > 0$ such that for all $s, t \in R$ with $|s-t| < \gamma$,

$$\eta(t)\ |\phi(s) - \phi(t)| < \varepsilon \quad .$$

Clearly, this implies that $\|\delta_h * \phi - \phi\| \leq \varepsilon$ for $0 < |h| < \gamma$, and we have proved that $\delta_h * \phi \to \phi$ in $BUC(\eta)$ whenever $\phi \in BUC(\eta)$. If $\phi \in BC_0(\eta)$, then by Lemma 2.1 and 2.2, $\delta_h * \phi \in BC_0(\eta)$, and by the preceding argument, $\delta_h * \phi \to \phi$ in $BC_0(\eta)$.

Next consider the case $\phi \in L^p(\eta)$, $1 \leq p < \infty$. If ϕ is continuous and has compact support, then $\delta_h * \phi \to \phi$ uniformly, and also in $L^p(\eta)$ as $h \to 0$. The set of functions of this type is dense in $L^p(\eta)$, so $\delta_h * \phi \to \phi$ for every $\phi \in L^p(\eta)$.

□

As a corollary we have the following lemma.

<u>Lemma 3.4</u>. <u>If</u> ϕ <u>is locally absolutely continuous with</u> $\phi' \in L^p(\eta)$, $1 \leq p < \infty$, <u>then</u> $\tau_h\phi - \phi \in L^p(\eta)$, <u>and</u> $h^{-1}(\tau_h\phi - \phi) \to \phi'$ <u>in</u> $L^p(\eta)$, <u>as</u> $h \to 0$. <u>Then the same statement is true with</u> $L^p(\eta)$ <u>replaced by</u> $BUC(\eta)$, <u>and by</u> $BC_0(\eta)$.

This follows directly from Lemma 3.3, because $h^{-1}(\tau_h\phi - \phi) = \delta_h * \phi'$.

We let $W^{1,p}(\eta)$, $1 \le p \le \infty$, be the set of locally absolutely continuous functions $\phi \in L^p(\eta)$ satisfying $\phi' \in L^p(\eta)$. Similarly, let $BUC^1(\eta)$ (and $BC_0^1(\eta)$) consist of those continuously differentiable functions $\phi \in BUC(\eta)$ (or $\phi \in BC_0(\eta)$) such that $\phi' \in BUC(\eta)$ (or $\phi' \in BC_0(\eta)$).

Lemma 3.5. Let $\mu \in M(\rho)$, *and* $\phi \in W^{1,p}(\eta)$, $1 \le p < \infty$. *Then* $\mu*\phi \in W^{1,p}(\eta)$, *and* $h^{-1}(\tau_h(\mu*\phi) - \mu*\phi) \to \mu*\phi'$ *in* $L^p(\eta)$ *as* $h \to 0$. *In particular,* $(\mu*\phi)' = \mu*\phi'$. *The same statements are true with* $W^{1,p}(\eta)$ *replaced by* $BUC^1(\eta)$ *and by* $BC_0^1(\eta)$, *and* $L^p(\eta)$ *replaced by* $BUC(\eta)$ *and by* $BC_0(\eta)$.

This follows again directly from Lemma 3.3, combined with Lemma 2.1.

Lemma 3.6. Let $a \in L^1(\rho)$ *with* $da \in M(\rho)$, *and let* $\phi \in L^p(\eta)$, $1 \le p < \infty$. *Then* $a*\phi \in W^{1,p}(\eta)$, *and* $h^{-1}(\tau_h(a*\phi) - a*\phi) \to da*\phi$ *in* $L^p(\eta)$ *as* $h \to 0$. *In particular,* $(a * \phi)' = da * \phi$. *The same statements are true with* $W^{1,p}(\eta)$ *replaced by* $BUC^1(\eta)$, *and by* $BC_0^1(\eta)$, *and* $L^p(\eta)$ *replaced by* $BUC(\eta)$ *and by* $BC_0(\eta)$.

Proof. Clearly it suffices to show that $h^{-1}(\tau_h(a*\phi) - a*\phi) \to da*\phi$ in $B(\eta)$ whenever $\phi \in B(\eta)$, where $B(\eta)$ is one of the spaces $L^p(\eta)$, $1 \le p < \infty$, $BUC(\eta)$ or $BC_0(\eta)$. By Lemma 3.3, $\delta_h * (da*\phi) \to da*\phi$ in $B(\eta)$ as $h \to 0$. However, $\delta_h * (da*\phi) = (\delta_h*da) * \phi$, and δ_h*da is almost everywhere defined by the function

$$\delta_h*da(t) = h^{-1}(a(t+h) - a(t)) \quad .$$

Thus,

$$\begin{aligned} \delta_h * (da*\phi) &= h^{-1}((\tau_h a) * \phi - a*\phi) \\ &= h^{-1}(\tau_h(a*\phi) - a*\phi) \quad , \end{aligned}$$

so $h^{-1}(\tau_h(a*\phi) - a*\phi) \to da*\phi$ in $B(\eta)$, as desired.

□

Lemma 3.7. $W^{1,p}(\eta) \subset BC_0(\eta)$ *for* $1 \le p < \infty$, *and* $W^{1,\infty}(\eta) \subset BUC(\eta)$.

Proof. Take $\phi \in W^{1,p}(\eta)$, $1 \le p \le \infty$, and use (2.3) to get

$$\|\tau_h\phi - \phi\| = \sup_{t\in R} \eta(t)|\phi(t+h) - \phi(t)|$$

$$\le \sup_{t\in R} \eta(t) \ |\int_t^{t+h}|\phi'(s)|ds|$$

$$\le \Big(\sup_{\substack{t\in R \\ |s-t|\le|h|}} \frac{\eta(t)}{\eta(s)}\Big)\Big(\sup_{t\in R} |\int_t^{t+h} \eta(s)|\phi'(s)|ds|\Big)$$

$$\le \Big(\sup_{|v|\le|h|} \rho(v)\Big)\Big(\sup_{t\in R} |\int_t^{t+h} \eta(s)|\phi'(s)|ds|\Big) \quad .$$

The first factor $\sup_{|v|\le|h|} \rho(v)$ is bounded as $h \to 0$, and the second factor

$$\sup_{t\in R} |\int_t^{t+h} \eta(s)|\phi'(s)|ds|$$

tends to zero as $h \to 0$, because $\eta\phi'$ belongs to the standard, non-weighted L^p-space over R. This proves that $\phi \in BUC(\eta)$.

Now suppose that $p < \infty$. Then by Lemma 2.5 and the preceding result, $\eta\phi$ is uniformly continuous and belongs to L^p (without weights), so $\eta(t)\phi(t) \to 0$ as $t \to \pm\infty$.

□

Remark 3.8. Lemmas 2.2 and 3.3 - 3.6 are also true when $p = \infty$, i.e. with $L^p(\eta)$ replaced by $L^\infty(\eta)$, and $W^{1,p}(\eta)$ replaced by $W^{1,\infty}(\eta)$, provided one throughout replaces the strong convergence with weak*-convergence. This follows from the fact that these lemmas are true in $L^1(\check{\eta})$, where

$$\check{\eta}(t) = (\eta(-t))^{-1} \quad (t \in R) \quad , \tag{3.8}$$

and that $L^\infty(\eta)$ can be identified with the dual of $L^1(\check{\eta})$ through the duality mapping

$$\langle\phi,\psi\rangle = \phi*\psi(0) \quad .$$

Observe that $\check{\eta}$ is an influence function dominated by ρ iff η is so.

4. A Modified Problem

We shall now turn to our study of (1.1), (1.2) or (1.2) - (1.5) in a fading memory space. Let $\prime$ be a dominating function or R^+, i.e. a positive, continuous function on R^+ satisfying

$$\rho(s+t) \le \rho(s)\rho(t) \quad (s,\, t \in R^+),\ \rho(0) = 1 \quad . \tag{4.1}$$

In addition, assume that

$$\rho_* = -\inf_{t>0} \frac{\log \rho(t)}{t} = -\lim_{t\to\infty} \frac{\log \rho(t)}{t} < \infty \quad , \tag{4.2}$$

(cf. [30, §2]), and suppose that

$$\rho(t)\exp(\rho_* t) \ \underline{\text{is nondecreasing}} \quad . \tag{4.3}$$

When (4.1) - (4.3) hold, one may extend ρ to a dominating function on R, as defined in Section 2, by putting

$$\rho(t) = \exp(-\rho_* t) \quad (t < 0) \quad , \tag{4.4}$$

and in the sequel we throughout assume that (4.4) holds (although other extensions are possible; cf. [31]. We denoted the subset of $M(\rho)$ than vanishes on $(-\infty,0)$ by $M(R^+;\rho)$.

In the sequel we shall not deal with (1.1), (1.2) or (1.2) - (1.5) directly, but rather with modified versions of these equations. The L^p case, i.e. the modifications of (1.2) - (1.5) is slightly simpler, so we treat this case first. In (1.2) - (1.5), let $\mu,\ \nu \in M(R^+;\rho;C^{n\times n})$, $f,\ g \in L^p(R^+;\eta;C^n)$, and $\phi \in L^p(R^-;\eta;C^n)$, where $1 \le p \le \infty$, and η is an influence function dominated by ρ. Redefine x and define y for $t \le 0$, and ϕ for $t > 0$ by

$$x(t) = 0 \quad (t \le 0) \quad , \tag{4.5}$$

$$y(t) = \begin{cases} 0 & (t \le -1) \quad , \\ (1+t)y_0 & (-1 < t \le 0) \quad , \end{cases} \tag{4.6}$$

$$\phi(t) = 0 \quad (t > 0) \quad . \tag{4.7}$$

Define

$$f_1(t) = \begin{cases} y(t) & (t \le 0) \quad , \\ \mu*\phi(t) + f(t) & (t > 0) \quad , \end{cases} \tag{4.8}$$

$$g_1(t) = \begin{cases} y'(t) & (t \le 0) \quad , \\ \nu*\phi(t) + g(t) & (t > 0) \quad . \end{cases} \tag{4.9}$$

Then ϕ, f_1, $g_1 \in L^p(R;\eta;C)$, with f_1 and g_1 depending linearly and continuously on ϕ, f, g and y_0, and (1.2) - (1.5) are transformed into

$$\mu * x(t) + f_1(t) = y(t) \quad (t \in R) \quad , \tag{4.10}$$

$$y'(t) = \nu * x(t) + g_1(t) \quad (t \in R) \quad , \tag{4.11}$$

with initial conditions (4.5), (4.6) (of course, (4.10) and (4.11) hold only almost everywhere). The fact that (4.10), (4.11) are equations on R rather than on R^+, and that all functions vanish for $t \leq -1$, make (4.10) and (4.11) easier to analyse than (1.2) - (1.5).

In the continuous case we use essentially the same transformation, but modify the functions close to zero in order to make all functions continuous. This time take μ, ν as above, $f, g \in BUC(R^+;\eta;C^n)$, and $\phi \in BUC(R^-;\eta;C^n)$. Define y by (1.3) to transform (1.1) into (1.3), (1.4). Redefine x and define y for $t < 0$ by

$$x(t) = \begin{cases} 0 & (t \leq -1) \quad , \\ (1+t)\phi(0) & (-1 < t \leq 0) \quad , \end{cases} \tag{4.12}$$

$$y(t) = \begin{cases} 0 & (t \leq -1) \quad , \\ (1+t)^2(\alpha t + \beta) & (-1 < t \leq 0) \quad , \end{cases} \tag{4.13}$$

where α and β are chosen so that $y(0) = \mu*\phi(0) + f(0)$ and $y'(0) = \nu*\phi(0) + g(0)$, i.e. $\beta = \mu*\phi(0) + f(0)$ and $\alpha = \nu*\phi(0) + g(0) - 2\beta$. Define

$$\bar{\phi}(t) = \begin{cases} \phi(t) - x(t) & (t \leq 0) \quad , \\ 0 & (t > 0) \quad , \end{cases} \tag{4(14)}$$

$$f_1(t) = \begin{cases} y(t) - \mu*x(t) & (t \leq 0) \quad , \\ f(t) + \mu*\bar{\phi}(t) & (t > 0) \quad , \end{cases} \tag{4.15}$$

$$g_1(t) = \begin{cases} y'(t) - \nu*x(t) & (t \leq 0) \quad , \\ g(t) + \nu*\bar{\phi}(t) & (t > 0) \quad . \end{cases} \tag{4.16}$$

Then $\bar{\phi}$, f_1, $g_1 \in BUC(R;\eta;C^n)$, with f_1 and g_1 depending linearly and continuously on ϕ, f and g, and (1.1), (1.2) are again transformed into (4.10), (4.11), but this time with the initial conditions (4.12), (4.13). If $f, g \in BC_0(R^+;\eta;C^n)$, then $f_1, g_1 \in BC_0(R;\eta;C^n)$.

We call μ atomic at zero, if

$$\det \mu(\{0\}) \neq 0 \quad , \tag{4.17}$$

i.e. μ has an invertible point mass at the origin. We assume throughout below that (4.17) holds, as one usually does when one studies a neutral equation (if (4.17) is violated, then the equation may become advanced rather than neutral). Thanks to the fact that our initial data vanish for $t < -1$, one can apply existence and uniqueness results for the case of a finite delay, to show that (4.10), (4.11) with initial conditions (4.5), (4.6) or (4.12), (4.13) has a unique solution on R (see [3] for the L^p-case and e.g. [18, p. 275] for the continuous case). However, in general (4.10), (4.11) also has solutions which do not satisfy the initial conditions, and these will play an important role in the sequel. We shall solve (4.10), (4.11) by using "resolvents" or "fundamental solutions". The resolvent which vanishes for $t < 0$ will give the solution of (4.10), (4.11) which satisfies the right initial condition, and the other resolvents will be used in our decomposition of solutions into stable and unstable components.

5. On the Resolvents

If one applies the Laplace transform to (4.10), (4.11), and solves for $\hat{x}$, $\hat{y}$, then one gets, at least formally,

$$\hat{x}(z) = D^{-1}(z)(\hat{g}_1(z) - z\hat{f}_1(z)) \quad , \tag{5.1}$$

$$\hat{y}(z) = \hat{\mu}(z)D^{-1}(z)\hat{g}_1(z) - \hat{\nu}(z)D^{-1}(z)\hat{f}_1(z) \quad , \tag{5.2}$$

where

$$D(z) = z\hat{\mu}(z) - \hat{\nu}(z) \quad . \tag{5.3}$$

Still proceeding formally, suppose that r is a function, locally of bounded variation, such that

$$\hat{r}(z) = D^{-1}(z) \quad . \tag{5.4}$$

Then the solution of (4.10), (4.11) should be given by

$$x = r*g_1 - dr*f_1 \quad , \tag{5.5}$$

$$y = \mu*r*g_1 - \nu*r*f_1 \quad . \tag{5.6}$$

In the sequel we make this formal argument precise.

We follow R. L. Wheeler [45] and G. S. Jordan and R. L. Wheeler [31), and define the determinant measure $\det \mu$ of μ by computing the formal determinant of μ, but replacing all pointwise multiplications by convolutions. More precisely, if $\mu = (\mu_{ij})$, then

$$\det \mu = \sum_{\tau \in S_n} \operatorname{sign}(\tau)\mu_{1\tau(1)} * \mu_{2\tau(2)} *\cdots* \mu_{n\tau(n)} \quad ,$$

where S_n is the group of permutations of $\{1,\ldots,n\}$. As $\mu \in M(\rho;C^{n\times n})$ is supported on R^+, we have $\det \mu \in M(\rho;C)$, and $\det \mu$ is supported on R^+. Split $\det \mu$ into its discrete, singular and absolutely continuous part

$$\det \mu = (\det \mu)_d + (\det \mu)_s + (\det \mu)_a \quad ,$$

just in the same way as in [31]. Finally, define

$$\Omega = \{\lambda > \rho_* \mid \inf_{\omega \in R} |(\det \mu)_d^{\wedge}(\lambda + i\omega)| > \|(\det \mu)_s\|_\lambda\} \quad , \tag{5.7}$$

where

$$\|(\det \mu)_s\|_\lambda = \int_0^\infty \exp(-\lambda t)\, d|(\det \mu)_s|(t) \quad . \tag{5.8}$$

Below we work most of the time in measure spaces different from our original measure space $M(\rho)$. Define

$$\eta_\lambda(t) = e^{-\lambda t} \quad (t \in R) \quad , \tag{5.9}$$

$$\eta_{\beta,\alpha}(t) = \begin{cases} e^{-\beta t} & (t \in R^-) \quad , \\ e^{-\alpha t} & (t \in R^+) \quad . \end{cases} \tag{5.10}$$

Then η_λ is a dominating function, and so is $\eta_{\beta,\alpha}$ if $\alpha \le \beta$. Define M_λ and $M_{\alpha,\beta}$ to be the measure spaces $M_\lambda = M(\eta_\lambda)$, and $M_{\alpha,\beta} = M(\eta_{\beta,\alpha})$. Observe that $M_{\alpha,\beta} = M_\alpha \cap M_\beta$, and that $M(R^+;\rho) \subset M_{\alpha,\beta}$ for all α,β satisfying $\rho_* \le \alpha \le \beta$. Let L^1_λ and $L^1_{\alpha,\beta}$ be the corresponding spaces of integrable functions, contained in M_λ and $M_{\alpha,\beta}$. As in (5.8), we denote norms in M_λ and L_λ by $\|\ \|_\lambda$, and norms in $M_{\alpha,\beta}$ and $L^1_{\alpha,\beta}$ by $\|\ \|_{\alpha,\beta}$.

In the sequel we throughout let δ stand for either the scalar-valued or the matrix-valued unit point mass at zero. We use I to denote the identity matrix in $C^{n\times n}$.

In our first theorem we construct solutions r_λ to (5.4).

<u>Theorem 5.1</u>. <u>Let</u> $\lambda \subset \Omega$, <u>and assume that</u> $\det D(z) \ne 0$ <u>on the line</u> $\mathrm{Re}\, z = \lambda$. <u>Then there exists a unique</u> $r_\lambda \in L^1_\lambda$ <u>with</u> $dr_\lambda \in M_\lambda$ <u>such that</u>

$$dr_\lambda * \mu - r_\lambda * \nu = \mu * dr_\lambda - \nu * r_\lambda = \delta \quad . \tag{5.11}$$

<u>Moreover,</u> $r_\lambda \in L^1_{\lambda,\lambda+\varepsilon}$ <u>and</u> $dr_\lambda \in M_{\lambda,\lambda+\varepsilon}$ <u>for some</u> $\varepsilon > 0$. <u>If</u> $\lambda = \rho_*$, <u>and</u>

$$\inf_{\omega\in R} |(\det \mu)^{\wedge}_d(\rho_* + i\omega)| > \int_0^\infty \rho(t)d|(\det \mu)_s|(t) \quad , \tag{5.12}$$

<u>then</u> $r_\lambda \in L^1(\rho)$ <u>and</u> $dr_\lambda \in M(\rho)$.

The proof given below is adapted from [31].

<u>Proof</u>. To get uniqueness it suffices to observe that (5.11) and Lemma 3.1 imply

$$z\hat{r}_\lambda(z)\hat{\mu}(z) - \hat{r}_\lambda(z)\hat{\nu}(z) = I \quad (\mathrm{Re}\, z = \lambda) \quad .$$

hence

$$\hat{r}_\lambda(z) = D^{-1}(z) \quad (\mathrm{Re}\, z = \lambda) \quad , \tag{5.13}$$

and that a function in L^1_λ is uniquely determined by its Laplace transform on the line $\mathrm{Re}\, z = \lambda$.

Suppose that we can find $\varepsilon > 0$, a function $r_\lambda \in L^1_{\lambda,\lambda+\varepsilon}$ such that (5.13) holds, and a measure $s_\lambda \in M_{\lambda,\lambda+\varepsilon}$ such that

$$\hat{s}_\lambda(z) = zD^{-1}(z) \quad (\mathrm{Re}\, z = \lambda) \quad . \tag{5.14}$$

Then, by Lemma 3.1, r_λ is locally of bounded variation, and $dr_\lambda = s_\lambda$. Define

$$t_\lambda = dr_\lambda * \mu - r_\lambda * \nu \quad .$$

Then, for $\mathrm{Re}\, z = \lambda$,

$$\hat{t}_\lambda(z) = z\hat{r}_\lambda(z)\hat{\mu}(z) - \hat{r}_\lambda(z)\hat{\nu}(z)$$

$$= D^{-1}(z)(z\hat{\mu}(z) - \hat{\nu}(z)) = I \quad ,$$

so $t_\lambda = \delta$, i.e. $dr_r * \mu - r_\lambda * \nu = \delta$. In the same way one shows that $\mu * dr_\lambda - \nu * r_\lambda = \delta$, so r_λ is the solution of our problem.

It remains to find $\varepsilon > 0$, $r_\lambda \in L^1_{\lambda,\lambda+\varepsilon}$ and $s_\lambda \in M_{\lambda,\lambda+\varepsilon}$ satisfying (5.13) and (5.14). As $\lambda \subset \Omega$, we have

$$\inf_{\omega \in R} |(\det \mu)_d^\wedge(\lambda + i\omega)| > \|(\det \mu)_s\|_\lambda \quad ,$$

and so by the uniform continuity of $(\det \mu)_d^\wedge(z)$ in $\operatorname{Re} z \geq \lambda$, we can find $\varepsilon > 0$ such that

(5.15) $$\inf_{\lambda \leq \operatorname{Re} z \leq \lambda+\varepsilon} |(\det \mu)_d^\wedge(z)| > \|(\det \mu)_s\|_\lambda \quad .$$

In particular,

$$\liminf_{\substack{|z| \to \infty \\ \lambda \leq \operatorname{Re} z \leq \lambda+\varepsilon}} |\det \hat{\mu}(z)| > 0 \quad ,$$

and as $(\det D(z))/z - \det \hat{\mu}(z) \to 0$ $(|z| \to \infty, \lambda \leq \operatorname{Re} z \leq \lambda+\varepsilon)$, we must have

$$\liminf_{\substack{|z| \to \infty \\ \lambda \leq \operatorname{Re} z \leq \lambda+\varepsilon}} |\det D(z)| > 0 \quad .$$

In the strip $\lambda < \operatorname{Re} z < \lambda+\varepsilon$ $\det D(z)$ is analytic, so it can have only finitely many zeros there. By decreasing the value of ε, if necessary, we may assume that

(5.16) $$\det D(z) \neq 0 \quad (\lambda \leq \operatorname{Re} z \leq \lambda+\varepsilon) \quad .$$

It follows from [13, Satz 7] and (5.15) that $(\det \mu)_d + (\det \mu)_s$ has an inverse q in $M_{\lambda,\lambda+\varepsilon}$, i.e. there exists a measure $q \in M_{\lambda,\lambda+\varepsilon}$ such that

(5.17) $$[(\det \mu)_d^\wedge(z) + (\det \mu)_s^\wedge(z)]\hat{q}(z) = 1$$

for $\lambda \leq \operatorname{Re} z \leq \lambda+\varepsilon$. Fix an arbitrary $\alpha < \lambda$, and define $e(t) = \exp(\alpha t)$ $(t \geq 0)$, $e(t) = 0$ $(t < 0)$. Then $e \in L^1_{\lambda,\lambda+\varepsilon}$, and $\hat{e}(z) = (z-\alpha)^{-1}$. Write (5.13) in the form

$$\hat{r}_\lambda(z) = D^{-1}(z) = \hat{e}(z)[\hat{\mu}(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))]^{-1}$$

$$= \frac{\hat{e}(z)\hat{q}(z) \operatorname{adj} [\hat{\mu}(z)+\hat{e}(z)(\alpha\hat{\mu}(z)-\hat{\nu}(z))]}{\hat{q}(z) \det [\hat{\mu}(z)+\hat{e}(z)(\alpha\hat{\mu}(z)-\hat{\nu}(z))]} \quad ,$$

where $\operatorname{adj}[\hat{\mu}(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))]$ is the adjoint of the matrix $\mu(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))$. This matrix is the transform of a measure in $M_{\lambda,\lambda+\epsilon}$, namely the matrix-valued measure one gets by taking the formal adjoint of $\mu + e*(\alpha\mu-\nu)$, replacing multiplications by convolutions. Thus, if we can invert the denumerator in (5.18), then (5.18) defines a measure in $M_{\lambda,\lambda+\epsilon}$.

Observe that $\det[\mu + e*(\alpha\mu-\nu)]$ has the same discrete and singular parts as $\det \mu$ (because $e*(\alpha\mu-\nu)$ is absolutely continuous). This, together with (5.17), implies that

$$\hat{q}(z)\det[\hat{\mu}(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))] = 1 + \hat{q}(z)\hat{h}(z) \quad ,$$

where h is the absolutely continuous part of $\det[\mu + e*(\alpha\mu-\nu)]$. Moreover, $1 + \hat{q}(z)\hat{h}(z) = (\det D(z))\hat{q}(z)\hat{e}(z) \neq 0$ for $\lambda < \operatorname{Re} z < \lambda+\epsilon$. Apply e.g. [30, Theorem 2.3] to get a function $d \in L^1_{\lambda,\lambda+\epsilon}$ such that

$$[1 + \hat{q}(z)\hat{h}(z)]^{-1} = 1 + \hat{d}(z) \quad (\lambda < \operatorname{Re} z < \lambda+\epsilon) \quad .$$

This means that (5.18) becomes

$$\hat{r}_\lambda(z) = \hat{e}(z)\hat{q}(z)(1+\hat{d}(z)) \operatorname{adj}[\hat{\mu}(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))] \quad . \tag{5.19}$$

This is a sum of products of transforms of measures in $M_{\lambda,\lambda+\epsilon}$, multiplied by the transform $e(z)$ of $e \in L^1_{\lambda,\lambda+\epsilon}$. *Thus, (5.19) defines* r_λ as an element of $L^1_{\lambda,\lambda+\epsilon}$.

To get a solution $s_\lambda \in M_{\lambda,\lambda+\epsilon}$ of (5.14) we multiply (5.19) by z, and observe that

$$z\hat{e}(z) = 1 + \alpha\hat{e}(z) \quad (\lambda < \operatorname{Re} z < \lambda+\epsilon) \quad .$$

Thus,

$$\hat{s}_\lambda(z) = (1 + \alpha\hat{e}(z))\hat{q}(z)(1 + \hat{d}(z))\operatorname{adj}[\hat{\mu}(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))] \quad ,$$

and this defines s_λ as an element of $M_{\lambda,\lambda+\epsilon}$.

Essentially the same argument gives the special claim $r_\lambda \in L^1(\rho)$ and $dr_\lambda \in M(\rho)$ when $\lambda = \rho_*$, *and (5.12) holds.*

□

Recall that μ is atomic at zero, if $\det \mu(\{0\}) \neq 0$.

<u>Theorem 5.2</u>. <u>Let</u> μ <u>be atomic at zero. Then there exists a constant</u> $d > \rho_*$ <u>such that</u> $[d,\infty) \subset \Omega$, <u>and</u> $\det D(z) \neq 0$ <u>for</u> $\operatorname{Re} z > d$. <u>Moreover,</u> r_d <u>vanishes for</u> $t < 0$, <u>and</u> $r_\lambda = r_d$ <u>for every</u> $\lambda > d$, <u>where</u> r_λ <u>and</u> r_d <u>are the resolvents constructed in Theorem 5.1</u>.

Proof. The proof of Theorem 5.2 is very similar to the proof of Theorem 5.1. The main difference is that we, this time, work in the space M_d^+ of measures M_d, vanishing on $(-\infty,0)$, and that we invert the determinant of $\mu + e*(\alpha\mu-\nu)$ in (5.18) in an elementary way, using a norm estimate, instead of [13, Satz 7].

Fix an arbitrary $\alpha < \rho_*$, define $e(t) = \exp(\alpha t)$ $(t \geq 0)$, $e(t) = 0$ $(t < 0)$, and write (5.13) in the form

$$\begin{aligned}\hat{r}_d(z) &= \hat{e}(z)\ [\hat{\mu}(z) + \hat{e}(z)(\alpha\hat{\mu}(z) - \hat{\nu}(z))]^{-1} \\ &= \hat{e}(z)\ \frac{\operatorname{adj}[\hat{\mu}(z)+\hat{e}(z)(\alpha\hat{\mu}(z)-\hat{\nu}(z))]}{\det[\hat{\mu}(z)+\hat{e}(z)(\alpha\hat{\mu}(z)-\hat{\nu}(z))]}\ .\end{aligned} \tag{5.20}$$

If we choose $d > \rho_*$, then $e, \mu, \nu \in M_d^+$, and if we can find an inverse to $\det(\mu + e*(\alpha\mu - \nu))$ in M_d^+ then (5.20) defines a solution r_d of (5.11) in M_d^+. As $M_d^+ \subset M_\lambda^+ \subset M_\lambda$ for every $\lambda \geq d$, the uniqueness of the solution of (5.11) in M_λ yields $r_\lambda = r_d$ for $\lambda \geq d$. Thus it only remains to find $d > \rho_*$ such that $[d,\infty) \subset \Omega$, $\det D(z) \neq 0$ for $\operatorname{Re} z \geq d$, and such that $\det(\mu + e*(\alpha\mu - \nu)$ has an inverse in M_d^+.

As μ is atomic at zero, and $\mu + e*(\alpha\mu - \nu)$ has the same discrete part as μ, the measure $\det(\mu + e*(\alpha\mu - \nu))$ has a point mass at the origin of size $\det \mu(\{0\}) \neq 0$. Define $a = \det \mu(\{0\})$, and put $q = a\delta - \det(\mu + e*(\alpha\mu - \nu))$. By Lebesque's dominated convergence theorem,

$$\|q\|_\lambda = \int_0^\infty \exp(-\lambda t)\, d|q|(t) \to 0 \quad (\lambda \to \infty) \quad ,$$

so we can find a constant $d > \rho_*$ such that $\|q\|_d < |a|$. As $\|q/a\| < 1$, the measure $a\delta - q = \det(\mu + e*(\alpha\mu - \nu))$ has a (convolution) inverse in M_d^+, which one gets by simply expanding $(a\delta - q)^{-1}$ into a (convolution) power series. Thus, there exists a measure $p \in M_d^+$ (i.e. $p = (a\delta - q)^{-1}$) such that $p * \det(\mu + e*(\alpha\mu - \nu)) = \delta$, and so (5.20) defines a solution $r_d \in M_d^+$ of (5.11). The Laplace transform of r_d converges absolutely for $\operatorname{Re} z \geq d$, so necessarily $\det D(z) \neq 0$ $(\operatorname{Re} z \geq d)$. That $[d,\infty) \subset \Omega$ follows from the fact that for $\lambda \geq d$,

$$\inf_{\omega\in R} |\widehat{(\det \mu)}_d(\lambda + i\omega)| - \|(\det \mu)_s\|_\lambda$$

$$> |a| - \int_{(0,\infty)} \exp(-\lambda t)\big(d|(\det \mu)_d|(t) + d|(\det \mu)_s|(t)\big)$$

$$> |a| - \|q\|_\lambda > |a| - \|q\|_d > 0 \quad .$$

□

6. Decomposing the Solutions

By applying Theorems 5.1 and 5.2, we get solutions to (4.10), (4.11) of the form (5.5), (5.6).

In Section 5 we worked with the dominating functions η_λ and $\eta_{\beta,\alpha}$, defined in (5.9), (5.10). The number λ in Theorem 5.1 determines the growth rate of the solution of (4.10), (4.11) that we obtain by using the resolvent r_λ. Below we will have to replace our original fading memory spaces by the spaces that one gets by choosing the influence function to be either η_λ or $\eta_{\beta,\alpha}$. Observe that $\tilde{\eta}_{\beta,\alpha} = \eta_{\alpha,\beta}$. If $\alpha < \beta$, the $\eta_{\alpha,\beta}$ is no longer a dominating function, but it is still an influence function dominated by $\eta_{\beta,\alpha}$. Also note that my (2.5), (2.6) and (4.4), $\eta(t) > \eta_\lambda(t)$ $(t > 0,\ \lambda > \rho_*)$.

When we say below that a solution x, y of (4.10), (4.11) belongs to "the appropriate fading memory space with influence function η_λ", we mean that if $f_1, g_1 \in BUC(\eta)$, then $x, y \in BUC(\eta_\lambda)$, if $f_1, g_1 \in BC_0(\eta)$, then $x, y \in BC_0(\eta_\lambda)$, and if $f_1, g_1 \in L^p(\eta)$, $1 \leq p \leq \infty$, then $x, y \in L^p(\eta_\lambda)$.

Lemma 6.1. *Let $r_\lambda \in L^1_\lambda$, with $dr \in M_\lambda$ be a solution of (5.11), and define*

(6.1) $$x_\lambda = r_\lambda * g_1 - dr_\lambda * f_1 \quad ,$$

(6.2) $$y_\lambda = \mu * r_\lambda * g_1 - \nu * r_\lambda * f_1 \quad .$$

Then x_λ, y_λ belong to the appropriate fading memory space with influence function η_λ, and x_λ, y_λ is a solution of (4.10), (4.11).

Actually, the solution x_λ, y_λ in Lemma 6.1 is unique, i.e. no other solution x, y of (4.10), (4.11) belongs to the appropriate fading memory space with influence function η_λ. However, we do not rely on this fact below, and we leave the uniqueness proof to the reader.

Proof. That x_λ, y_λ belong to the right space follows from Lemma 2.1. Use (5.11), (6.1), (6.2) and Lemmas 3.5 and 3.6 to get

$$\mu * x_\lambda + f_1 = \mu * r_\lambda * g_1 + (\delta - \mu * dr_\lambda) * f_1 = y \quad ,$$

$$\nu * x_\lambda + g_1 = (\delta + \nu * r_\lambda) * g_1 - \nu * dr_\lambda * f_1 =$$

$$= \mu * dr_\lambda * g_1 - \nu * dr_\lambda * f_1 = y' \quad .$$

This means that x_λ, y_λ satisfy (4.10), (4.11).

□

Lemma 6.2. In addition to the assumption of Lemma 6.1, suppose that r_λ vanishes on $(-\infty,0)$. Then x_λ, y_λ satisfy $x_\lambda(t) = x(t)$, $y_\lambda(t) = y(t)$ $(t \leq 0)$, where x and y are given by either (4.5), (4.6) or (4.12), (4.13).

Proof. By (4.15), (4.16) (or (4.8), (4.9)), (5.11), (6.1), Lemmas 3.5 and 3.6, and the fact that r_λ vanishes for $t < 0$, we have on the interval $(-\infty,0]$,

$$x_\lambda = r_\lambda * g_1 - dr_\lambda * f_1$$

$$= r_\lambda * (y' - \nu * x) - dr_\lambda * (y - \mu * x)$$

$$= r_\lambda * y' - dr_\lambda * y + (dr_\lambda * \mu - r_\lambda * \nu) * x = x \quad ,$$

so $x_\lambda(t) = x(t)$ for $t \leq 0$ (almost everywhere in the L^p-case). By (4.10), for $t \leq 0$, $y_\lambda = \mu * x_\lambda + f_1 = \mu * x + f_1 = y$, so also $y_\lambda(t) = y(t)$ for $t \leq 0$.

□

Lemma 6.3. Let r_α and r_β be as in Lemma 6.1, with $\alpha \leq \beta$. Then $x_{\alpha,\beta} = x_\beta - x_\alpha$ and $y_{\alpha,\beta} = y_\beta - y_\alpha$ belong to the appropriate fading memory space with influence function $\eta_{\alpha,\beta}$, and $x_{\alpha,\beta}$ $y_{\alpha,\beta}$ is a solution of the homogeneous equations (1.7), (1.8).

This is a direct consequence of Lemma 6.1 and the fact that $\eta_{\alpha,\beta} = \min\{\eta_\alpha,\eta_\beta\}$.

Lemma 6.4. Let $[\alpha,\beta] \subset \Omega$, with $\det D(z) \neq 0$ on the lines $\mathrm{Re}\, z = \alpha$ and $\mathrm{Re}\, z = \beta$. Then $\det D(z)$ has at most finitely many zeros z_j, $1 \leq j \leq m$, of finite order k_j in the strip $\alpha < \mathrm{Re}\, z < \beta$. Moreover, $x_{\alpha,\beta}$ and $y_{\alpha,\beta}$ in Lemma 6.3 are of the form

(6.3) $$x_{\alpha,\beta}(t) = \sum_{j=1}^{m} p_j(t)e^{z_j t} \quad ,$$

(6.4) $$y_{\alpha,\beta}(t) = \sum_{j=1}^{m} q_j(t)e^{z_j t} \quad ,$$

<u>where</u> p_j <u>and</u> q_j <u>are polynomials of degree at most</u> $k_j - 1$. <u>In particular, if</u> $\det D(z) \neq 0$ <u>in the strip</u> $\alpha \leq \mathrm{Re}\, z \leq \beta$, <u>then</u> $x_{\alpha,\beta} = y_{\alpha,\beta} = 0$.

The proofs of (6.3) and (6.4) are completely similar, so we prove only (6.3) below.

<u>Proof</u>. Recall that the resolvents r_λ in Theorem 5.1 satisfy $r_\lambda \in L^1_{\lambda,\lambda+\varepsilon}$ and $dr_\lambda \in M_{\lambda,\lambda+\varepsilon}$ for some $\varepsilon > 0$. In particular, $r_\lambda \in L^1_{\lambda_1}$ and $dr_\lambda \in M_{\lambda_1}$ for every λ_1, $\lambda \leq \lambda_1 \leq \lambda_1 + \varepsilon$. By the uniqueness of the solution r_{λ_1} in L_{λ_1}, $r_\lambda = r_{\lambda_1}$ for $\lambda \leq \lambda_1 \leq \lambda+\varepsilon$. The interval $[\alpha,\beta]$ is compact, so if $\det D(z) \neq 0$ for $\alpha \leq \mathrm{Re}\, z \leq \beta$, then we must have $r_\lambda = r_\alpha = r_\beta$ for every $\lambda \in [\alpha,\beta]$. In particular, $x_{\alpha,\beta} = (r_\beta - r_\alpha)*g_1 - (dr_\beta - dr_\alpha)*d_1 = 0$, and we have proved the last statement of Lemma 6.4.

That $\det D(z)$ can have at most finitely many zeros of finite order in a strip $\lambda \leq \mathrm{Re}\, z \leq \lambda+\varepsilon$ was established in the proof of Theorem 5.1 (the lines following (5.15)). By the compactness of the interval $[\alpha,\beta]$, the same is true in the whole strip $\alpha \leq \mathrm{Re}\, z \leq \beta$.

To get (6.3) in the general case we go back to the proof of Theorem 5.1. Fix λ, $\alpha < \lambda < \beta$. Analogously to (5.15), we can find λ_1 and λ_2, with $\alpha < \lambda_1 < \lambda < \lambda_2 < \beta$ such that

$$\inf_{\lambda_1 \leq \mathrm{Re}\, z \leq \lambda_2} |(\det \hat{\mu})_d(z)| > \|(\det \mu)_s\|_{\lambda_1} \quad .$$

Then the solution q of (5.17) satisfies $q \in M_{\lambda_1,\lambda_2}$. If necessary, modify the values of λ_1 and λ_2 slightly to ensure that

(6.5) $$\det D(z) \neq 0 \quad (\mathrm{Re}\, z = \lambda_j,\ j = 1,2) \quad .$$

Recall that $f_1, g_1 \in L^1_{\lambda_1,\lambda_2}$ (since $\lambda_1 > \rho_*$). By (5.13) and (6.1), on the lines $\mathrm{Re}\, z = \lambda_j$ $(j = 1,2)$,

(6.6) $$\hat{x}_{\lambda_1}(z) = D^{-1}(z)\hat{g}_1(z) - zD^{-1}(z)\hat{f}_1(z) \quad .$$

The same argument as in Section 5 shows that the right hand side of (6.6) is a vector of (extended) locally analytic functions on the maximal ideal space $\{z \in C \mid \lambda_1 < \mathrm{Re}\, z < \lambda_2\}$ of $L^1_{\lambda_1,\lambda_2}$, with a finite number of poles at the zeros z_j of $\det D(z)$ that lie in the strip $\lambda_1 < \mathrm{Re}\, z_j < \lambda_2$ (see [30, Definitions 3.2 and 7.1]). The pole at z_j is of order at most k_j. Apply [30, Theorem 3.6] to the components of this vector to get a function $a \in L^1_{\lambda_1,\lambda_2}(C^n)$ and constants $\beta_{l,j}$ $(1 \le l \le k_j)$ such that

$$\hat{x}_{\lambda_1}(z) = \sum_{\lambda_1 < \mathrm{Re}\, z_j < \lambda_2} \beta_{l,j}(z-z_j)^{-l} + \hat{a}(z) \tag{6.7}$$

for $\mathrm{Re}\, z = \lambda_j$ $(j = 1,2)$. Take the inverse transform of (6.7) for $j = 1$ and $j = 2$ to get for almost all t,

$$x_{\lambda_1}(t) = \begin{cases} a(t) & (t < 0) \;, \\ a(t) + \displaystyle\sum_{\lambda_1 < \mathrm{Re}\, z_j < \lambda_2} \beta_{l,j} \frac{t^{l-1}}{(l-1)!} e^{z_j t} & (t \ge 0) \;, \end{cases} \tag{6.7}$$

and

$$x_{\lambda_2}(t) = \begin{cases} a(t) - \displaystyle\sum_{\lambda_1 < \mathrm{Re}\, z_j < \lambda_2} \beta_{l,j} \frac{t^{l-1}}{(l-1)!} e^{z_j t} & (t < 0) \;, \\ a(t) & (t \ge 0) \;. \end{cases} \tag{6.8}$$

Subtract (6.7) from (6.8) to get

$$x_{\lambda_2}(t) - x_{\lambda_1}(t) = \sum_{\lambda_1 < \mathrm{Re}\, z_j < \lambda_2} \beta_{l,j} \frac{t^{l-1}}{(l-1)!} e^{z_j t} \quad (t \in R) \;.$$

The final conclusin now follows in the same way as in the case when $\det D(z)$ has no zeros in the strip $\alpha < \mathrm{Re}\, z < \beta$.

$\square$

We are finally ready to state and prove our main result. We give two formulations, one for the continuous case, and one for the L^p-case.

Theorem 6.5. Let $[\alpha,\beta]\ \Omega$, and assume that $\det D(z) \neq 0$ on the lines $\mathrm{Re}\, z = \alpha$ and $\mathrm{Re}\, z = \beta$. Then the continuous solutions x of (1.1), (1.2) can be written as a unique sum $x = x_S + x_C + x_U$, where x_S is a solution of (1.1), x_C and x_U are solutions of (1.6), x_S and x_U satisfy

$$|x_S(t)| = O(\exp(\alpha t)) \quad (t \to \infty) \quad , \tag{6.9}$$

$$|x_U(t)| = \begin{cases} O(\exp((\beta+\varepsilon)t)) & (t \to -\infty) \quad , \\ O(\exp(dt) & (t \to \infty) \end{cases} \tag{6.10}$$

(where d is the constant in Theorem 5.2, and $\varepsilon > 0$), and x_C is an exponential polynomial

$$x_C(t) = \sum_{j=1}^{m} p_j(t) e^{z_j t} \quad . \tag{6.11}$$

Here z_j $(1 \leq j \leq m)$ are the zeros of $\det D(z)$ in the strip $\alpha < \mathrm{Re}\, z < \beta$, and p_j are polynomials of degree at most one less than the order of the zero z_j. In particular, if $\det D(z) \neq 0$ for $\alpha \leq \mathrm{Re}\, z \leq \beta$, then $x_C = 0$. If $\beta \geq d$, then $x_U = 0$. If $\alpha = \rho_*$ and (5.12) holds, then

$$x_S \in BUC(\eta) \quad , \tag{6.12}$$

and if moreover $\phi \in BC_0(R^-;\eta)$, $f, g \in BC_0(R^+;\eta)$, then $x_S \in BC_0(\eta)$.

Theorem 6.6. Let $[\alpha,\beta] \subset \Omega$, and assume that $\det D(z) \neq 0$ on the lines $\mathrm{Re}\, z = \alpha$ and $\mathrm{Re}\, z = \beta$. Then the L^p-solution $(1 \leq p \leq \infty)$ of (1.2) - (1.5) can be written as a unique sum $x = x_S + x_C + x_U$, $y = y_S + y_C + y_U$, where x_S, y_S is a solution of (1.3), (1.4), x_C, y_C and x_U, y_U are solutions of (1.7), (1.8), and the components satisfy the following conditions. Let d be the constant of Theorem 5.2, let ε be given as in Theorem 5.1, with $\lambda = \beta$, and define

$$\eta_S(t) = \begin{cases} \eta(t) & (t \leq 0) \quad , \\ \exp(-\alpha t) & (t > 0) \quad , \end{cases} \tag{6.13}$$

$$\eta_U(t) = \begin{cases} \exp(-(\beta+\varepsilon)t) & (t < 0) \; , \\ \exp(-dt) & (t > 0) \; . \end{cases} \tag{6.14}$$

Then $x_S, y_S \in L^p(\eta_S)$, *and* $x_U, y_U \in L^p(\eta_U)$. *The central components* x_C *and* y_C *are exponential polynomials*

$$x_C = \sum_{j=1}^{m} p_j(t) e^{z_j t} \; , \tag{6.15}$$

$$y_C(t) = \sum_{j=1}^{m} q_j(t)\, e^{z_j t} \; , \tag{6.16}$$

where z_j $(1 \le j \le m)$ *are the zeros of* $\det D(z)$ *in the strip* $\alpha < \operatorname{Re} z < \beta$, *and* p_j, q_j *are the polynomials of degree at most one less than the order of the zero* z_j. *In particular, if* $\det D(z) \neq 0$ *for* $\alpha \le \operatorname{Re} z \le \beta$, *then* $x_C = y_C = 0$. *If* $\beta > d$, *then* $x_U = y_U = 0$. *If* $\alpha = \rho_*$ *and* (5.12) *holds, then* $x_S, y_S \in L^p(\eta)$.

Clearly, one gets Theorem 6.5 from Theorem 6.6 by substituting throughout BUC or BC_0 for L^p. The proofs of the two theorems are completely similar, so we give a combined proof.

Proof. Define

$$x_S = \phi + x_\alpha, \; x_C = x_\beta - x_\alpha, \; x_U = x_d - x_\beta \; , \tag{6.17}$$

$$y_S = y_\alpha, \; y_C = y_\beta - y_\alpha, \; y_U = y_d - y_\beta \; , \tag{6.18}$$

where the functions on the right hand side are defined as in Lemma 6.1, with the resolvents constructed in Theorem 5.1 and 5.2. Then $x = x_S + x_C + x_U = x_d + \phi$, and $y = y_S + y_C + y_U = y_d$, so by Lemmas 6.1 and 6.2, and by the construction in Section 4, x, y is the solution of (1.2) - (1.5) (or (1.2 - (1.4) in the continuous case). That the components satisfy the right equations and have the right growth rates follows from (6.17), (6.18), Lemma 2.1, Theorem 5.1 and Lemmas 6.1 and 6.3. Lemma 6.4 tells us that x_C, y_C are of the given form. Finally, if $\beta > d$, then by (6.1), (6.2), (6.17), (6.18) and Theorem 5.2, $x_U = y_U = 0$.

That the decomposition in Theorem 6.5 and 6.6 is unique follows from the fact that the different components have different exponential growth rate. Let $x = \bar{x}_S + \bar{x}_C + \bar{x}_U$, $y = \bar{y}_S + \bar{y}_C + \bar{y}_U$ be another decomposition, and define $\tilde{x}_S = x_S - \bar{x}_S$, $\tilde{y}_S = y_S - \bar{y}_S$, etc. Then $\tilde{x} = \tilde{x}_S + \tilde{x}_C + \tilde{x}_U$, $\tilde{y} = \tilde{y}_S + \tilde{y}_C + \tilde{y}_U$ is a solution of (1.2) - (1.5), with $\phi = f = g = \gamma = 0$, so by the uniqueness of the solution of (1.2) - (1.5), $x = y = 0$. Thus, $\tilde{x}_U = -(\tilde{x}_S + \tilde{x}_C)$, $\tilde{y}_U = -(\tilde{y}_S + \tilde{y}_C)$, and by using the growth estimates that we have on the different components, one can show that $\tilde{x}_U, \tilde{y}_U \in L^1(\eta_{\lambda_1})$, where η_{λ_1} is the function defined in (5.9), with $\lambda_1 = \beta + \varepsilon/2$, and ε is the constant in (6.14). As $\tilde{x}_U, \tilde{y}_U$ is a solution of (1.7), (1.8), we must have

$$D(z)(\tilde{x}_U)\hat{\ }(z) = 0 \quad (\text{Re } z = \lambda_1) \quad .$$

But $\det D(z) \neq 0$ on the line $\text{Re } z = \lambda_1$ (because r_{λ_1} in Theorem 5.1 belongs to $L^1(\eta_{\lambda_1})$), so necessarily $(\tilde{x}_U)\hat{\ }(z) = 0$ $(\text{Re } z = \lambda_1)$, i.e. $\tilde{x}_U = 0$. By (1.7), also $\tilde{y}_U = 0$. This means that $\tilde{x}_C = -\tilde{x}_S$, $\tilde{y}_C = \tilde{y}_S$, and by comparing the growth rates of the left and right hand sides we find that $\tilde{x}_C = \tilde{y}_C = 0$. Clearly then, also $\tilde{x}_S = \tilde{y}_S = 0$, and the decomposition is unique.

$\square$

<u>Remark 6.7</u>. If the initial data belong to $W^{1,p}(\eta)$, $1 \le p \le \infty$, then by Lemmas 3.5 and 3.6, and the way in which we constructed and decomposed the solutions, x and its components belong to $W^{1,p}$, and y and its components belong to $W^{2,p}$, with the appropriate influence functions. Here $W^{2,p}(\eta)$ consists of functions y such that y, y' and y'' all belong to $L^p(\eta)$. When $p = \infty$, this tells us that Lipschitz continuity is preserved.

7. <u>A Semigroup Interpretation</u>

Our decomposition theorems can be interpreted in a semigroup setting. For simplicity we discuss only the homogeneous equation

$$\frac{d}{dt}(\mu * x(t)) = \nu * x(t) \quad (t \in R^+) \quad , \tag{7.1}$$

with initial condition

$$x(t) = \phi(t) \quad (t \in R^-) \tag{7.2}$$

in the continuous case, and the corresponding homogeneous equations

(7.3) $$\mu * x(t) = y(t) \quad (t \in R^+) \quad ,$$

(7.4) $$y'(t) = \nu * x(t) \quad (t \in R^+) \quad ,$$

with initial condition (7.2) and

(7.5) $$y(0) = y_0$$

in the L^p-case. The semigroup generated by (7.1), (7.2) is somewhat different from the semigroup generated by (7.2) - (7.5), so we discuss the two case separately.

First consider the continuous case. Let x be the solution of (7.1), (7.2), let Δ_t be the combined translation and restriction operator defined in (2.20), and define operator $T(t)$ for $t \in R^+$ by

(7.6) $$T(t)\phi = \Delta_t x \quad .$$

Then $T(t)$ maps $BUC(R^-;\eta;C^n)$ into itself, and it has the semigroup property $T(s+t) = T(s)T(t)$ $(s, t \in R^+)$. By the definition of $BUC(\eta)$, the translation operator τ_t is strongly continuous in $BUC(\eta)$, and this implies that $T(t)$ is strongly continuous in $BUC(R^-;\eta)$. Thus, $T(t)$ becomes a strongly continuous semigroup.

We claim that the domain $D(A)$ of the generator A of $T(t)$ is the set

(7.7) $$D(A) = \{\phi \in BUC^1(R^-;\eta) \mid \mu * \phi'(0) = \nu * \phi(0)\}$$

and that

(7.8) $$A\phi = \phi' - \quad (\phi \subset D(A)) \quad .$$

Here $\phi'(t)$ is the ordinary derivative of $\phi(t)$ for $t < 0$, and $\phi'(0)$ is the left-derivative of ϕ at zero,

By the definition of the generator of a semigroup [25, p. 302],

(7.9) $$A\phi = \lim_{h \to 0+} h^{-1}(\Delta_h x - \phi) = \lim_{h \to 0+} h^{-1}(\Delta_h x - \Delta_0 x)$$

with $D(A)$ consisting of those $\phi \in BUC(R^-;\eta)$ for which this limit exists in $BUC(R^-;\eta)$. In particular, (7.9) implies that x has a continuous right-derivative $x^{(+)}$ for $t \le 0$, hence $x(t) = \phi(t)$ has a continuous derivative $\phi'(t) = x^{(+)}(t)$ for $t \le 0$ (where $\phi'(0)$ stands for a left-derivative), and that $A\phi = \phi' \in BUC(R^-;\eta)$.

Conversely, suppose that $\phi \in BUC(R^-;\eta)$ has a derivative $\phi' \in BUC(\bar{R}^-;\eta)$, and that x has a right-derivative $x^{(+)}(0)$ at zero, with $x^{(+)}(0) = \phi'(0)$ (where $\phi'(0)$ again is the left-derivative of ϕ at zero). Then x has a right-derivative $x^{(+)} \in BUC(R^-;\eta)$, and by the half-line version of Lemma 3.4, the limit in (7.9) exists and equals ϕ'. In particular, $\phi \in D(A)$. This shows that (7.8) holds, and that $D(A)$ consists of exactly those $\phi \in BUC(R^-;\eta)$ satisfying $\phi' \in BUC(R^-;\eta)$ for which $\phi'(0) = x^{(+)}(0)$. To get (7.7) one must identify the condition $\phi'(0) = x^{(+)}(0)$ with the condition $\mu*\phi'(0) = \nu*\phi(0)$. We leave this step to the reader, as it is essentially the same argument as in the case of finite delay (cf. [18, Thm. 10.1, p. 307]).

With the aid of Theorem 6.5 one can decompose $BUC(R^-;\eta;C^n)$ into three invariant subspaces, $BUC(R^-;\eta;C^n) = S \oplus C \oplus U$, where S is a "stable" subspace, C is a "central" subspace, and U is an "unstable" subspace.

Theorem 7.1. Let $[\alpha,\beta] \subset \Omega$, and assume that $\det D(z) \neq 0$ on the lines $\operatorname{Re} z = \alpha$ and $\operatorname{Re} z = \beta$. Then $BUC(R^-;\eta;C^n)$ can be decomposed in a unique way into subspaces S, C and U, with the following properties. The subspaces C and U are contained in $BC_0(R^-;\eta;C^n)$. Let $T_S(t)$, $T_C(t)$ and $T_U(t)$ be the restrictions of $T(t)$ to S, C and U. Then $T_C(t)$ and $T_U(t)$ can be extended to groups on C and U, and $T_S(t)$ and $T_U(t)$ satisfy

$$(7.10)\qquad |T_S(t)| = O(\exp(\alpha t)) \quad (t \to \infty) \quad ,$$

$$(7.11)\qquad |T_U(t)| = \begin{cases} O(\exp((\beta+\varepsilon)t)) & (t \to -\infty) \\ O(\exp(dt) & (t \to \infty) \end{cases} \quad ,$$

where d is the constant in Theorem 5.2, and $\varepsilon > 0$. The subspace C is finite dimensional, and it is spanned by functions ϕ of the form

$$(7.12)\qquad \phi(t) = \sum_{j=1}^{m} p_j(t)e^{z_j t} \quad (t \le 0) \quad ,$$

where z_j $(1 \le j \le m)$ are the zeros of $\det D(z)$ in the strip $\alpha < \operatorname{Re} z < \beta$, and p_j are polynomials of degree at most one less than the order of the zero z_j. In particular, if $\det D(z) \neq 0$ for $\alpha \le \operatorname{Re} z \le \beta$, then $C = \{0\}$. If $\beta > d$, then $U = \{0\}$. Finally, *if* $\alpha = \rho_*$, $\phi \in BC_0(R^-;\eta;C^n)$, *and* η *has the relaxation property, then*

(7.13) $$T_S(t)\phi = o(\exp(\rho_* t)) \quad (t \to \infty) \quad .$$

For a version of Theorem 7.1 with finite delay (and with the singular part of μ identically zero), see [22, Theorem 4.1 and 4.2]. The dimension of C is actually equal to the sum of the orders of the zeros of $\det D(z)$ in $\alpha < \operatorname{Re} z < \beta$; see [36].

Proof. We shall use Theorem 6.5 to define projections P_S, P_C and P_U of $BUC(R^-;\eta)$ onto S, C and U. For every $\phi \in BUC(R^-;\eta)$, let x be the solution of (7.1), (7.2), split it into $x_S + x_C + x_U$ as in Theorem 6.5, and define $P_S\psi = \Delta_0 x_S$, $P_C\phi = \Delta_0 x_C$, and $P_U\psi = \Delta_0 x_U$. The functions f_1 and g_1 defined in (4.15) - (4.16) are continuous, linear functions of ϕ, and the functions $\Delta_0 x_S$, $\Delta_0 x_C$ and $\Delta_0 x_U$ depend linearly and continuously on f_1 and g_1. This means that P_S, P_C and P_U are continuous, linear operators. They are projection operators, because x_S, x_C and x_U all satisfy (7.1), and the decomposition in Theorem 6.5 is unique. For the same reason they commute with $T(t)$ for $t > 0$. By (7.2), $P_S + P_C + P_U$ is the identity operator. Thus, defining S, C and U to be the ranges of P_S, P_C and P_U, we find that S, C and U are closed, invariant subspaces of $BUC(R^-;\eta)$ with $BUC(R^-;\eta) = S \oplus C \oplus U$. As x_C and x_U satisfy (1.6), we can define $T_C(t)$ and $T_U(t)$ also for negative t, and $T_C(t)$ and $T_U(t)$ become groups. That T_S and T_U satisfy the growth properties (7.10), (7.11) and (7.13) (if necessary, redefine η so that $\eta(t) = \exp(-\rho_* t)$ $(t > 0)$ to get (7.13)) follows from (6.17), Lemmas 2.2, 2.4 and 6.1, and Theorem 5.1. That C is spanned by functions ϕ of the form (7.12) follows from (6.11), and clearly C is finite dimensional. If $\beta > d$, then by Theorem 6.5, $U = \{0\}$. Finally, the uniqueness of the decomposition in Theorem 7.1 is a consequence of the uniqueness of the decomposition in Theorem 6.5.

□

Now consider the L^p-case. Let x, y be a solution of (7.2) - (7.5), and define the operator $Q(t)$ on $C^n \times L^p(R^-;\eta;C)$ by

(7.14) $$Q(t)(y_0,\phi) = (y(t),\Delta_t x) \quad .$$

Then $Q(t)$ maps $C^n \times L^p(R^-;\eta;C)$ into itself, it has the semigroup property $Q(s+t) = Q(s)Q(t)$ $(s,\ t \in R^+)$, and it is strongly continuous if $p < \infty$. In the sequel we therefore restrict the values of p to $1 \leq p < \infty$. One argues as in [3] to show that the domain $D(B)$ of the generator B of Q is given by

$$\{(y,\phi) \in C^n \times W^{1,p}(R^-;\eta;C) \mid \mu*\phi(0) = y\} \tag{7.15}$$

and that

$$B(y,\phi) = (\nu*\phi,\ \phi') \ . \tag{7.16}$$

Using Theorem 6.6 one can again decompose $C^n \times L^p(R^-;\eta;C)$ into three invariant subspaces, just as in Theorem 7.1. The result that one gets is very similar to Theorem 7.1, and its proof is almost identical to the proof of Theorem 7.1. Therefore, we leave the formulation and the proof of Theorem 7.2 to the reader.

8. Some Additional Comments

In Section 4 we started with a dominating function ρ on R^+ satisfying (4.3), extended it to R, and let η be an influence function dominated by ρ. For instance, if $\rho(t) = (1+t)^\gamma$ $(t \geq 0)$ for some positive constant γ, then we take $\rho(t) = 1$ $(t < 0)$, and we can choose

$$\eta(t) = \tilde{\rho}(t) = \begin{cases} (1+|t|)^{-\gamma} & (t < 0) \ , \\ 1 & (t \geq 0) \ . \end{cases} \tag{8.1}$$

This influence function tends to zero as $t \to -\infty$ (but not exponentially), and it has the relaxation property.

On the other hand, one could also follow the spirit of [41] and [44] and start with a dominating function ρ on R^-, assume that it satisfies a condition similar to (4.3), extend it to all of R, and choose $\eta = \rho$. Of course, then (4.4) is no longer true, but that causes only minor modifications in the main theorems (it affects the growth rates of the stable components in the case when $\alpha = \rho_*$). A more serious problem with this approach is that one cannot obtain growth rates for η anywhere close to those in (8.1), because by (3.1), if a submultiplicative function tends to zero at infinity, then it does so with exponential rate.

We have assumed throughout that μ is atomic at zero. This assumption is used essentially only to assure forward existence and uniqueness for the solutions of (1.1), (1.2) or (1.2) - (1.5). Even without it the technique used here yields forward existence in the stable and central subspaces, and backward existence in the central and unstable subspaces. There is an example in [3] of an equation for which one has forward existence and uniqueness without μ being atomic at zero, and our main theorems could be modified to apply to that equation.

One can sharpen Theorem 5.1 and subsequent results slightly by appearling to [30, Section 8] rather than to [13, Satz 7]. This permits one to replace $\|(\det \mu)_s\|_\lambda$ in (5.7) and $\int_0^\infty \rho(t)d|(\det \mu)_s|(t)$ in (5.12) by the spectral radii

$$\lim_{n\to\infty} \|(\det \mu)_s^{*n}\|_\lambda^{1/n}$$

and

$$\lim_{n\to\infty} [\int_0^\infty \rho(t)d|(\det \mu)_s^{*n}|(t)]^{1/n} ,$$

respectively, where $(\det \mu)_s^{*n}$ stands for the n-fold convolution of $(\det \mu)_s$ with itself. For more details, see [30, Section 8].

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: #2275	2. GOVT ACCESSION NO.: AD-A110
3. RECIPIENT'S CATALOG NUMBER: 484	
4. TITLE (and Subtitle): On a Neutral Functional Differential Equation in a Fading Memory Space	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Olof Staffans	8. CONTRACT OR GRANT NUMBER(s): DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS: U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709	12. REPORT DATE: September 1981
	13. NUMBER OF PAGES: 41
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Neutral functional differential equation, fading memory space, history space, phase space, exponential dichotomy, hyperbolicity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study the linear autonomous, neutral system of functional differential equations

$$(*)\quad \frac{d}{dt}(\mu * x(t) + f(t)) = \nu * x(t) + g(t) \quad (t \geq 0),$$
$$x(t) = \phi(t) \quad (t \leq 0)$$

in a fading memory space. Here μ and ν are matrix-valued measures supported on $[0,\infty)$, finite with respect to a weight function, and f, g and ϕ are

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ABSTRACT (continued)

C^n-valued continuous or locally integrable functions, bounded with respect to a fading memory norm. We give conditions which imply that solutions of (*) can be decomposed into a stable part and an unstable part. These conditions are of frequency domain type. We do not need the usual assumption that the singular part of μ vanishes. Our results can be used to decompose the semigroup generated by (*) into a stable part and an unstable part.

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